# Chapter 8 Digital Modulation in an Additive White Gaussian Noise Baseband Channel (V)

# The Optimum Detector (1/12)

- The received vector $y$ consists of two vectors. The first vector is $s_m$; the second vector is $n$

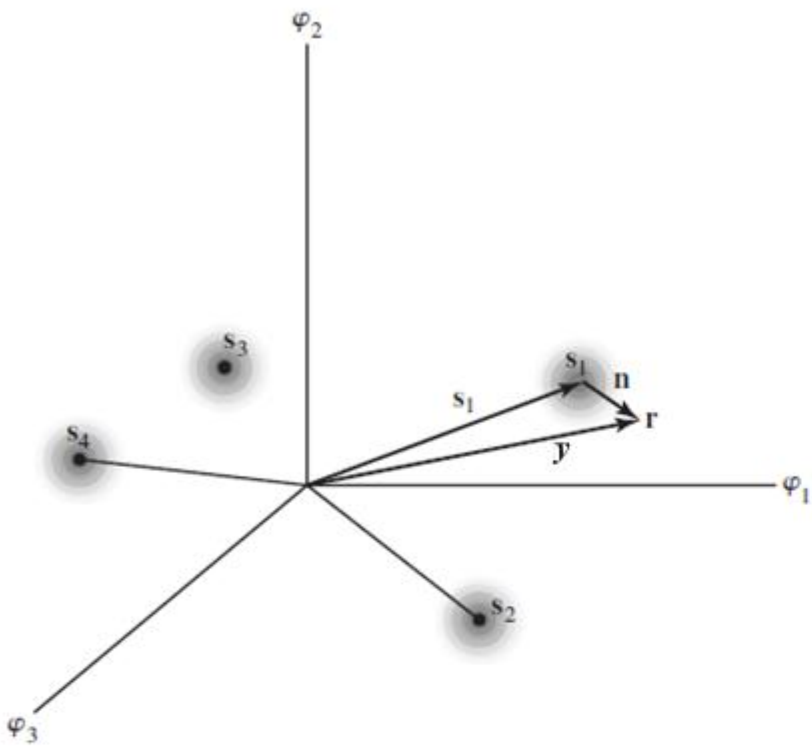


**Figure 8.43** Signal constellation, noise cloud and received vector for $N = 3$ and $M = 4$. It is assumed that $s_1$ is transmitted.

# The Optimum Detector (2/12)

- We wish to design a signal detector that makes a decision on the transmitted signal in each signal interval based on the observation of the vector $y$ in each interval, such that the probability of a correct decision is maximized
- We consider a decision rule based on the computation of the *posterior probabilities* defined as

$$P(\text{signal } \boldsymbol{s}_m \text{ was transmitted} \mid y), \quad m = 1, 2, \ldots, M,$$

which we abbreviate as $P(\boldsymbol{s}_m \mid y)$

- The decision criterion is based on selecting the signal corresponding to the maximum of the set of posterior probabilities $\{P(\boldsymbol{s}_m \mid y)\}$. We show this criterion maximizes the probability of a correct decision at the end of this section

# The Optimum Detector (3/12)

- It is clear that in the absence of any received information $y$, the best decision is to choose the signal $\boldsymbol{s}_m$ that has the highest prior probability $P(\boldsymbol{s}_m)$
- After receiving the information $y$, the prior probabilities $P(\boldsymbol{s}_m)$ are replaced with the posterior (conditional) probabilities $P(\boldsymbol{s}_m|y)$, and the receiver chooses the $\boldsymbol{s}_m$ that maximizes $P(\boldsymbol{s}_m|y)$. This decision criterion is called the *maximum a posteriori probability* (MAP) criterion
- We express the posterior probabilities as

$$P(\boldsymbol{s}_m|y) = f(y|\boldsymbol{s}_m)P(\boldsymbol{s}_m)/f(y), \qquad (8.4.49)$$

where $f(y|\boldsymbol{s}_m)$ is the conditional PDF of the observed vector given $\boldsymbol{s}_m$

# The Optimum Detector (4/12)

- The denominator of Eq. (8.4.49) may be expressed as

$$f(\mathbf{y})=\sum_{m=1}^{M} f(\mathbf{y}\,|\,\mathbf{s}_m)P(\mathbf{s}_m) \qquad (8.4.50)$$

- From Eqs. (8.4.49) and (8.4.50), we observe that the computation of the posterior probabilities $P(\mathbf{s}_m\,|\,\mathbf{y})$ requires knowledge of the *a priori* probabilities $P(\mathbf{s}_m)$ and the conditional PDF's $f(\mathbf{y}\,|\,\mathbf{s}_m)$ for $m=1,2,\ldots,M$. Note $f(\mathbf{y})$ is irrelevant with $m$
- When the $M$ signals are equally probable *a priori*, *i.e.*, $P(\mathbf{s}_m)=1/M$ for all $M$, the decision rule based on finding the signal that maximizes $P(\mathbf{s}_m\,|\,\mathbf{y})$ is equivalent to finding the signal that maximizes $f(\mathbf{y}\,|\,\mathbf{s}_m)$

# The Optimum Detector (5/12)

- The PDF $f(y \mid \boldsymbol{s}_m)$, or any monotonic function of it, is usually called the likelihood function. The decision criterion based on the maximum of $f(y \mid \boldsymbol{s}_m)$ over the $M$ signals is called the *maximum-likelihood (ML) criterion*
- A detector based on the MAP criterion and one that is based on the ML criterion make the same decisions, as long as the *a priori* probabilities $P(\boldsymbol{s}_m)$ are all equal
- We may work with the natural logarithm of $f(y \mid \boldsymbol{s}_m)$, which is a monotonic function. Thus,

$$\ln f(y \mid \boldsymbol{s}_m) = \tfrac{-N}{2}\ln(\pi N_0) - \tfrac{1}{N_0}\sum_{k=1}^{N}(y_k - s_{mk})^2$$

# The Optimum Detector (6/12)

- The maximum of ln $f(y \mid \boldsymbol{s}_m)$ over $\boldsymbol{s}_m$ is equivalent to finding the signal $\boldsymbol{s}_m$ that minimize the Euclidean distance

$$D(y, \boldsymbol{s}_m) = \sum_{k=1}^{N} (y_k - s_{mk})^2$$

- For the AWGN channel, the decision rule based on ML criterion reduce to finding the signal $\boldsymbol{s}_m$ that is closest in distance to the received signal vector $y$. We refer to this decision rule as *minimum distance detection*
- Expanding $D(y, \boldsymbol{s}_m)$, we have

$$D(y, \boldsymbol{s}_m) = \sum_{k=1}^{N} y_k^{\ 2} - 2\sum_{k=1}^{N} y_k s_{mk} + \sum_{k=1}^{N} s_{mk}^{\ 2}$$

$$= |y|^2 - 2y \cdot \boldsymbol{s}_m + |\boldsymbol{s}_m|^2, \ m = 1, 2, \ldots, M$$

# The Optimum Detector (7/12)

- The term $|y|^2$ is common to all decision metrics; hence, it may be ignored. Thus, the detection problem is transferred to maximize the metric

$$D'(y,\mathbf{s}_m)=-2y\cdot\mathbf{s}_m+|\mathbf{s}_m|^2,\ m=1,2,\ldots,M$$

- Note the *posterior probability metrics* is

$$PM(y,\mathbf{s}_m)=f(y|\mathbf{s}_m)P(\mathbf{s}_m)$$

- **Example 8.4.7.** Consider the case of binary PAM signals in which the two possible signal points are $s_1=-s_2=\sqrt{\mathcal{E}_b}$, where $\mathcal{E}_b$ is the energy per bit. The prior probabilities are $P(s_1)$ and $P(s_2)$. Determine the optimum MAP detector when the transmitted signal is corrupted with AWGN

# The Optimum Detector (8/12)

- **Example 8.4.7. (Cont'd)** The received signal vector (which is one dimensional) for binary PAM is

$$y = \pm\sqrt{\mathcal{E}_b} + n,$$

where $n$ is a zero-mean Gaussian random variable with a variance $\sigma_n^2 = N_0/2$.

- The conditional PDF's $f(y \mid s_m)$ for the two signals are

$$f(y \mid s_1) = \frac{1}{\sqrt{2\pi\sigma_n^2}} e^{-(y-\sqrt{\mathcal{E}_b})^2/2\sigma_n^2}$$

and

$$f(y \mid s_2) = \frac{1}{\sqrt{2\pi\sigma_n^2}} e^{-(y+\sqrt{\mathcal{E}_b})^2/2\sigma_n^2}$$

# The Optimum Detector (9/12)

- **Example 8.4.7. (Cont'd)** The posterior probability metrics are

$$PM(y,s_1)=f(y\mid s_1)P(s_1)$$

and

$$PM(y,s_2)=f(y\mid s_2)P(s_2)$$

- If $PM(y,s_1)>PM(y,s_2)$, we select $s_1$ as the transmitted signal; otherwise, we select $s_2$. Or, alternatively, if $\frac{PM(y,s_1)}{PM(y,s_2)}>1$, we select $s_1$
- But

$$\frac{PM(y,s_1)}{PM(y,s_2)}=\frac{P(s_1)}{P(s_2)}\exp\left\{[(y+\sqrt{\mathcal{E}_b})^2-(y-\sqrt{\mathcal{E}_b})^2]/2\sigma_n{}^2\right\}$$

# The Optimum Detector (10/12)

- **Example 8.4.7. (Cont'd)** Thus, we have

$$\frac{(y+\sqrt{\mathcal{E}_b})^2-(y-\sqrt{\mathcal{E}_b})^2}{2\sigma_n^2} \underset{s_2}{\overset{s_1}{\gtrless}} \ln\frac{P(s_2)}{P(s_1)},$$

  or, equivalently,

$$y \underset{s_2}{\overset{s_1}{\gtrless}} \frac{N_0}{4\sqrt{\mathcal{E}_b}} \ln\frac{P(s_2)}{P(s_1)},$$

- This is the final form for the optimum detector. We note that this is exactly the same detection rule obtained for binary antipodal signal with minimum bit error probability
- In the case of unequal prior probabilities, it is also necessary to know $N_0$ and $\mathcal{E}_b$. The threshold is zero if equally probable

# The Optimum Detector (11/12)

- We want to show that the MAP criterion is optimal in another way
- Let us denote by $R_m$ the correct region in the $N$-dimensional space for which the signal $s_m(t)$ was transmitted and the vector $\mathbf{y}=(y_1,y_2,\ldots,y_N)$ is received
- The probability of a decision error given that $s_m(t)$ was transmitted is

$$P(e\,|\,\mathbf{s}_m)=\int_{R_m{}^c} f(\mathbf{y}\,|\,\mathbf{s}_m)\,d\mathbf{y},$$

where $R_m{}^c$ is the complement of $R_m$

# The Optimum Detector (12/12)

- The average probability of error is

$$
\begin{aligned}
P_M &= \sum_{m=1}^{M} P(\mathbf{s}_m) P(e \mid \mathbf{s}_m) \\
&= \sum_{m=1}^{M} P(\mathbf{s}_m) \int_{R_m{}^c} f(\mathbf{y} \mid \mathbf{s}_m) d\mathbf{y} \\
&= \sum_{m=1}^{M} P(\mathbf{s}_m) \left[ 1 - \int_{R_m} f(\mathbf{y} \mid \mathbf{s}_m) d\mathbf{y} \right] \\
&= 1 - \sum_{m=1}^{M} \int_{R_m} P(\mathbf{s}_m) f(\mathbf{y} \mid \mathbf{s}_m) d\mathbf{y} \qquad \mathbf{(*)}
\end{aligned}
$$

- For the MAP criterion, when the $M$ signals are not equally probable, the average probability of error is

$$P_M = 1 - \sum_{m=1}^{M} \int_{R_m} P(\mathbf{s}_m \mid \mathbf{y}) f(\mathbf{y}) d\mathbf{y}$$

- Eq. (*) is a minimum when the points that are to be included in each particular region $R_m$ are those with largest posterior probabilities

# Probability of Error for *M*-ary Pulse Amplitude Modulation (1/7)

- Recall that binary PAM signals are antipodal signals. The probability of error of the optimum detector for equally probable binary PAM signals is

$$P_2 = Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right), \qquad (8.5.1)$$

where $Q(x)$ is Gaussian Q-function, $\mathcal{E}_b$ is the signal energy per bit, and $N_0/2$, is the power spectral density of the AWGN

- Note $2\mathcal{E}_b/N_0$ is the output SNR from the matched filter (and correlation-type) demodulator
- $\mathcal{E}_b/N_0$ is usually called the signal-to-noise ratio per bit or SNR/bit

# Probability of Error for *M*-ary Pulse Amplitude Modulation (2/7)

- The probability of error may be expressed in terms of the distance between the two signals $s_1$ and $s_2$. From Fig. 8.7, we observe that the two signals are separated by the distance $d_{12}=2\sqrt{\mathcal{E}_b}$
- Substituting $\mathcal{E}_b=d_{12}^2/4$ into Eq. (8.5.1), we obtain

$$P_2 = Q\left(\sqrt{\frac{d_{12}^2}{2N_0}}\right).$$

This expression illustrates the dependence of the error probability on the distance between the two signal points

# Probability of Error for *M*-ary Pulse Amplitude Modulation (3/7)

- In the case of *M*-ary PAM, the input to the detector is

$$y = s_m + n$$

where $s_m$ denotes the $m$th transmitted amplitude level, and $n$ is a Gaussian random variable with zero mean and variance $\sigma_n^2 = N_0/2$

- A decision is made in favor of the amplitude level that is closest to $y$

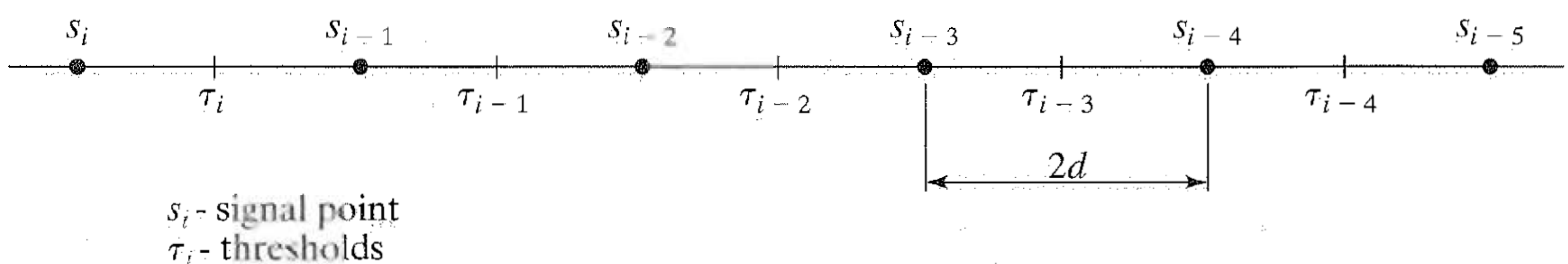


**Figure 8.44** Placement of thresholds at midpoints of successive amplitude levels.

# Probability of Error for *M*-ary Pulse Amplitude Modulation (4/7)

- On the basis that all amplitude levels are equally likely *a priori*, the average probability of a symbol error is simply the probability that the noise variable $n$ exceeds in magnitude one-half of the distance between levels
- However, when either one of the two most outer levels $\pm(M\text{-}1)$ is transmitted, an error can occur in one direction only. Thus, we have

$$\begin{aligned} P_M &= \tfrac{M-1}{M} P(|y - s_m| > d) \\ &= \tfrac{M-1}{M} \frac{2}{\sqrt{\pi N_0}} \int_d^{\infty} e^{-x^2/N_0} dx \\ &= \tfrac{M-1}{M} \frac{2}{\sqrt{2\pi}} \int_{\sqrt{2d^2/N_0}}^{\infty} e^{-x^2/2} dx \\ &= \tfrac{2(M-1)}{M} Q(\sqrt{2d^2/N_0}), \end{aligned}$$

where $2d$ is the distance between adjacent signal points

# Probability of Error for *M*-ary Pulse Amplitude Modulation (5/7)

- Recall that the average energy per symbol $\mathcal{E}_{av}$ can be represented as

$$\mathcal{E}_{av}=d^2(M^2\text{-}1)/3.$$

The average probability of error is expressed as

$$P_M=\tfrac{2(M-1)}{M}Q\left(\sqrt{6\mathcal{E}_{av}/(M^2-1)N_0}\right),$$

Since the average transmitted signal energy $\mathcal{E}_{av}=TP_{av}$, where $P_{av}$ is the average transmitted power, $P_M$ may also be expressed as a function of $P_{av}$

- Since each symbol carries $k=\log_2 M$ bits of information, the average energy per bit $\mathcal{E}_{bav}$ is given by $\mathcal{E}_{av}/k$. $P_M$ can be written as

$$P_M=\tfrac{2(M-1)}{M}Q(\sqrt{6(\log_2 M)\mathcal{E}_{bav}/(M^2-1)N_0}),$$

# Probability of Error for *M*-ary Pulse Amplitude Modulation (6/7)

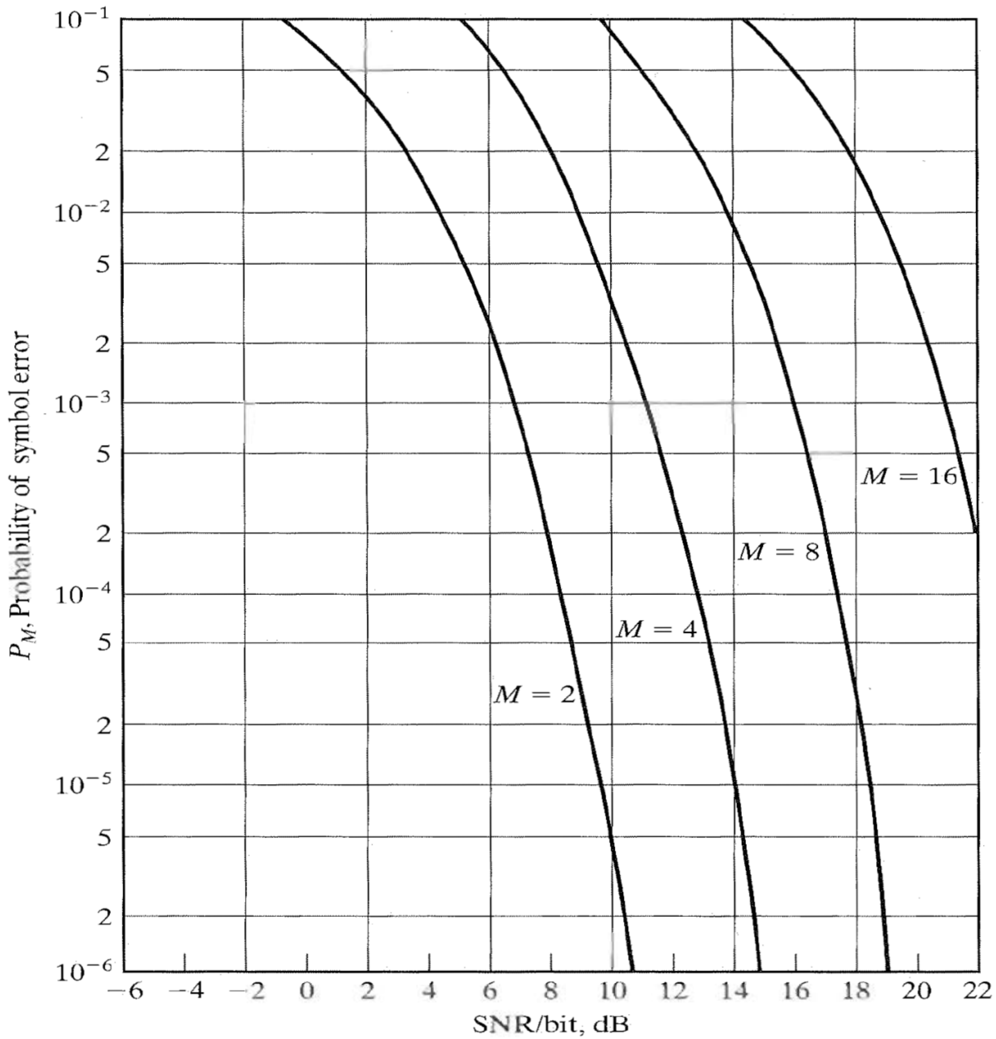


**Figure 8.45** Probability of a symbol error for PAM.

# Probability of Error for *M*-ary Pulse Amplitude Modulation (7/7)

- **Example 8.5.1.** Using Fig. 8.45, determine (approximately) the SNR/bit required to achieve a symbol error probability of $P_M = 10^{-6}$ for $M=2$, $M=4$, and $M=8$
- From observation of Fig. 8.45, we know that the required SNR/bit (approximately) as follows:

1. 10.5 dB for $M=2$ (1 bit/symbol)
2. 14.8 dB for M=4 (2 bits/symbol)
3. 19.2 dB for M=8 (3 bits/symbol)

- For small values of *M*, each additional bit requires an increase of bit energy by a little over 4 dB
- For large values of *M*, each additional bit requires an increase of bit energy by around 6 dB

# Probability of Error for *M*-ary Orthogonal Signals (1/7)

- Consider the probability of error of $M$-ary orthogonal PPM signaling over an AWGN channel.
- For equal energy orthogonal signals, the optimum detector selects the signal resulting in the largest cross correlation between the received vector $y$ and each of the $M$ possible transmitted signal vectors $\{\mathbf{s}_m\}$, *i.e.*,

$$C(\mathbf{y},\mathbf{s}_m)=\mathbf{y}\bullet\mathbf{s}_m=\sum_{k=1}^{N} y_k s_{mk}, \qquad m=1,2,\ldots,M$$

- Suppose that the signal $\mathbf{s}_1$ is transmitted. Then the vector at the input to the detector is

$$\mathbf{y}=(\sqrt{\mathcal{E}_s}+n_1,n_2,n_3,\ldots,n_N),$$

$n_1,n_2,\ \ldots,n_N$ are zero-mean, mutually independent Gaussian random variables with equal variance $\sigma_n^2=N_0/2$

# Probability of Error for *M*-ary Orthogonal Signals (2/7)

- Assume $N=M$ for simplicity, thus

$$C(y,s_1)=\sqrt{\mathcal{E}_s}(\sqrt{\mathcal{E}_s}+n_1);$$
$$C(y,s_2)=\sqrt{\mathcal{E}_s}\,n_2;$$
$$\vdots$$
$$C(y,s_M)=\sqrt{\mathcal{E}_s}\,n_M;$$

- Note the scale factor $\sqrt{\mathcal{E}_s}$ may be eliminated from the correlator outputs by dividing each output by $\sqrt{\mathcal{E}_s}$. The PDF of the first correlator output is

$$f(y_1)=\frac{1}{\sqrt{\pi N_0}}e^{-(y_1-\sqrt{\mathcal{E}_s})^2/N_0}$$

and the PDF's of the other $M$-1 correlator outputs are

$$f(y_m)=\frac{1}{\sqrt{\pi N_0}}e^{-y_m{}^2/N_0}, \qquad m=2,3,\ldots,M$$

# Probability of Error for *M*-ary Orthogonal Signals (3/7)

- The probability of a correct decision means the probability that $y_1$ is larger than each of the other $M$-1 correlator outputs $n_2, n_3, \ldots, n_M$. This probability may be expressed as

$$P_c = \int_{-\infty}^{\infty} P(n_2 < y_1, n_3 < y_1, \ldots, n_M < y_1 \mid y_1) f(y_1) dy_1,$$

where $P(n_2 < y_1, n_3 < y_1, \ldots, n_M < y_1 \mid y_1)$ denotes the joint probability that $n_2, n_3, \ldots, n_M$ are all less than $y_1$, conditioned on any given $y_1$

- Since the $\{y_m\}$ are statistically independent, the joint probability factors into a product of $M$-1 marginal probabilities of the form

$$P(n_m < y_1 \mid y_1) = \int_{-\infty}^{y_1} f(y_m) dy_m, \qquad m = 2,3,\ldots, M$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\sqrt{2y_1^2/N_0}} e^{-t^2/2} dt = 1 - Q\left(\sqrt{\frac{2y_1^2}{N_0}}\right)$$

# Probability of Error for *M*-ary Orthogonal Signals (4/7)

- These probabilities are identical for $m=2,3,\ldots,M$; hence, the joint probability under consideration is

$$P_c = \int_{-\infty}^{\infty} \left[1-Q\left(\sqrt{\frac{2y_1^2}{N_0}}\right)\right]^{M-1} f(y_1)dy_1$$

and the probability of a *k*-bit symbol error is

$$P_M = 1 - P_c$$

- Therefore,

$$P_M = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \{1-[1-Q(x)]^{M-1}\} e^{-(x-\sqrt{2\mathcal{E}_s/N_0})^2/2} dx$$

Since all the $M$ signals are equally likely, the expression for $P_M$ is the average probability of a symbol error

- $P_M$ can also be represented in terms of the SNR/bit, $\mathcal{E}_b/N_0$, by replacing $\mathcal{E}_s$ with $k\mathcal{E}_b$

# Probability of Error for *M*-ary Orthogonal Signals (5/7)

- Assume orthogonal signal sets are with equal energy and the distance between every pair of signals is the same. If $\boldsymbol{s}_1$ is transmitted, there are $M$-1 other signals to which an error symbol can be made. These wrongly detected symbols are all with the same probability
- The number of error patterns which are resulted from an error of $i$ bits out of the $k$ bits is $\binom{k}{i}$. Since all signals are the same distance from $\boldsymbol{s}_1$, the conditional probability of a symbol error with $i$ bits in error is

$$\frac{\binom{k}{i}}{M-1}$$

# Probability of Error for *M*-ary Orthogonal Signals (6/7)

- So the average number of bit error given a symbol error is

$$\sum_{i=1}^{k} i\binom{k}{i}/(M-1) = \frac{1}{M-1}\sum_{i=1}^{k}\frac{k!}{(i-1)!(k-i)!}$$

$$= \frac{1}{M-1}\sum_{i'=0}^{k-1}\frac{k\cdot(k-1)!}{i'!(k-i'-1)!}$$

$$= \frac{k2^{k-1}}{M-1}$$

- The probability of bit error given a symbol in error is

$$\frac{1}{k}\frac{k2^{k-1}}{M-1} = \frac{2^{k-1}}{2^k-1}.$$

Thus, we have

$$P_b = \frac{2^{k-1}}{2^k-1}P_M.$$

# Probability of Error for *M*-ary Orthogonal Signals (7/7)

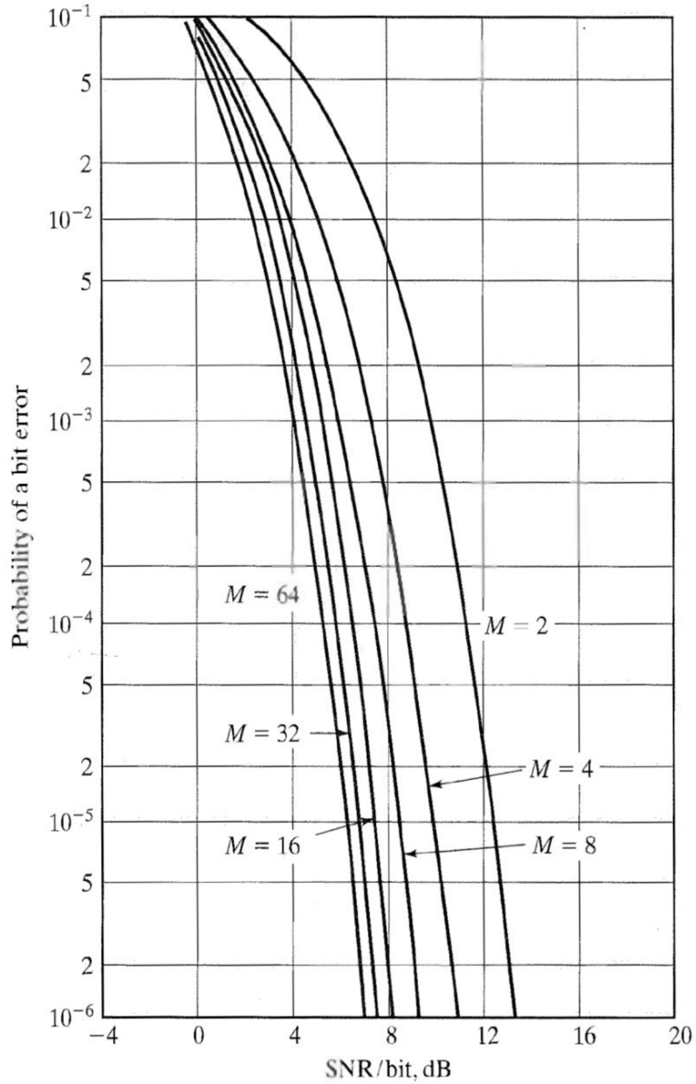


**Figure 8.46** Probability of a bit error for optimum detection of orthogonal signals.