# Chapter 8 Digital Modulation in an Additive White Gaussian Noise Baseband Channel (IV)

# *M*-ary Pulse Modulation (1/3)

- We consider the simultaneous transmission of multiple bits by using more than two signal waveforms
- The binary information sequence is subdivided into blocks of $k$ bits, called symbols, and each block (or symbol) is represented by one of $M=2^k$ signal waveforms, each of duration $T$. This type of modulation is called *M*-ary modulation
- The signaling (symbol) rate, $R_s$, as the number of signals (or symbols) transmitted per second. Clearly,

$$R_s = \frac{1}{T}$$

# $M$-ary Pulse Modulation (2/3)

- Each signal carries $k=\log_2 M$ bits of information, the bit rate is given by

$$R_b = kR_s = \frac{k}{T}$$

- The bit interval is

$$T_b = \frac{1}{R_b} = \frac{T}{k}$$

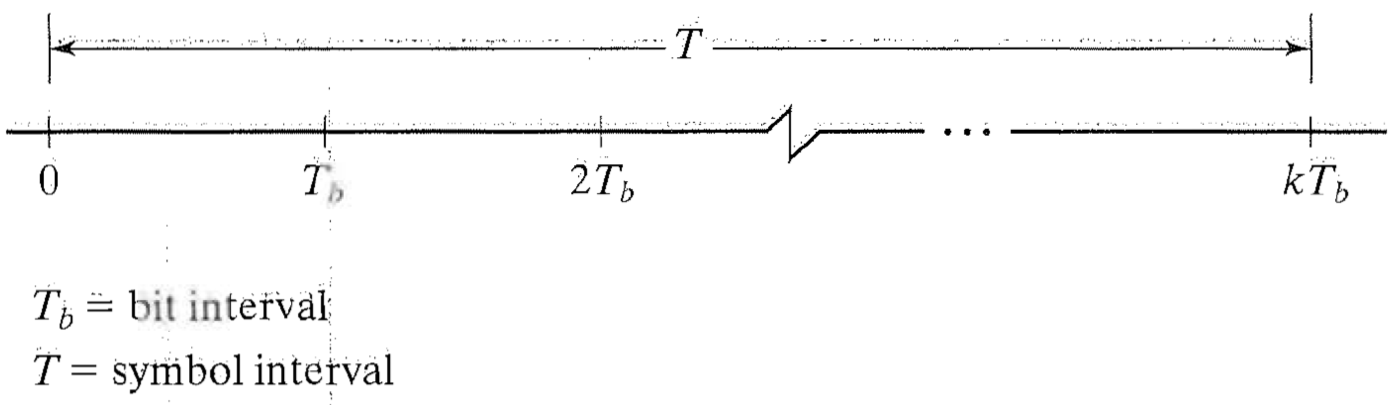


**Figure 8.27** Relationship between the symbol interval and the bit interval.

# *M*-ary Pulse Modulation (3/3)

- The *M* signal waveforms may be one-dimensional or multi-dimensional
- The one-dimensional *M*-ary signals are a generalization of the binary PAM (antipodal) signals. The multi-dimensional signals are a generalization of the binary PPM (orthogonal) signals

# $M$-ary Pulse Amplitude Modulation (1/5)

- The $k$-bit symbols are used to select $M=2^k$ signal amplitudes. The $M$-ary PAM signal waveforms may be expressed as

$$s_m(t) = A_m g_T(t), \quad 0 \le t \le T, \quad m = 1,2,\ldots,M$$
$$= s_m \psi(t), \quad 0 \le t \le T, \quad m = 1,2,\ldots,M$$

- All $M$ signal waveforms have the same pulse shape. Hence, they are one-dimensional signals. Note that $s_m = A_m\sqrt{T}$

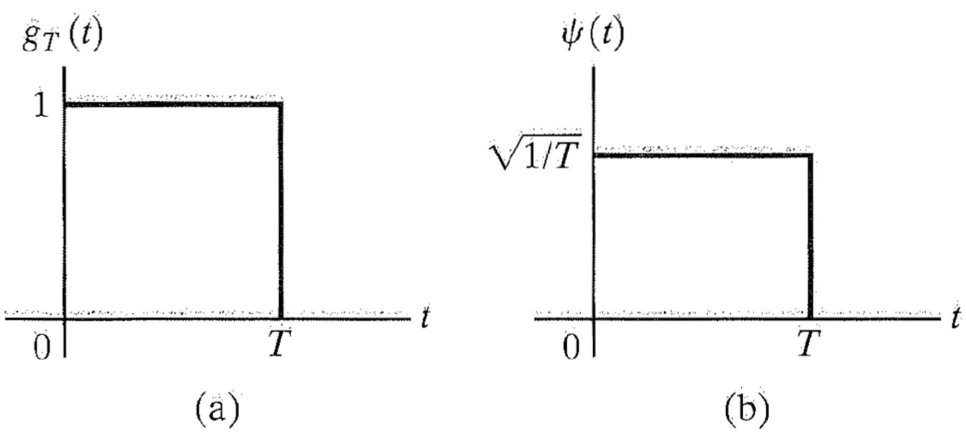


**Figure 8.28** Rectangular pulse $g_T(t)$ and basis function $\psi(t)$ for $M$-ary PAM.

# *M*-ary Pulse Amplitude Modulation (2/5)

- An important feature of these PAM signals is that they have different energies. That is,

$$\mathcal{E}_m = \int_0^T s_m^2(t)dt = s_m^2 \int_0^T \psi^2(t)dt = s_m^2 = A_m^2 T$$

- Assuming that all *k*-bit symbols are equally probable, the average energy of the transmitted signals is

$$\mathcal{E}_{av} = \frac{1}{M}\sum_{m=1}^{M} \mathcal{E}_m = \frac{T}{M}\sum_{m=1}^{M} A_m^2$$

- In order to minimize the average transmitted energy and to avoid transmitting signals with a DC component, we want to select the *M* signal amplitudes to be symmetric about the origin and equally spaced. That is,

$$A_m = (2m-1-M)A, \quad m = 1,2,\ldots,M$$

# *M*-ary Pulse Amplitude Modulation (3/5)

- The corresponding average energy, assuming that all *k*-bit symbols are equally probable, is

$$\mathcal{E}_{av} = \frac{A^2 T}{M} \sum_{m=1}^{M} (2m-1-M)^2$$
$$= A^2 T (M^2 - 1)/3$$

- The corresponding signal constellation point of the *M*-ary PAM signals are given as

$$s_m = A_m \sqrt{T}$$
$$= A\sqrt{T}(2m-1-M), \quad m = 1,2,\ldots,M$$

- It is convenient to define the distance parameter $d$ as $d = A\sqrt{T}$, so that

$$s_m = (2m-1-M)d, \quad m = 1,2,\ldots,M$$

# $M$-ary Pulse Amplitude Modulation (4/5)

- The signal constellation point diagram is shown in Fig. 8.29. The distance between two adjacent signal points is $2d$
- **Example 8.4.1.** Sketch the signal waveforms for $M$=4 PAM and determine the average transmitted signal energy.
- The average energy, based on equally probable signals, is

$$\mathcal{E}_{av}=5A^2T=5d^2,$$

where $d^2=A^2T$ by definition

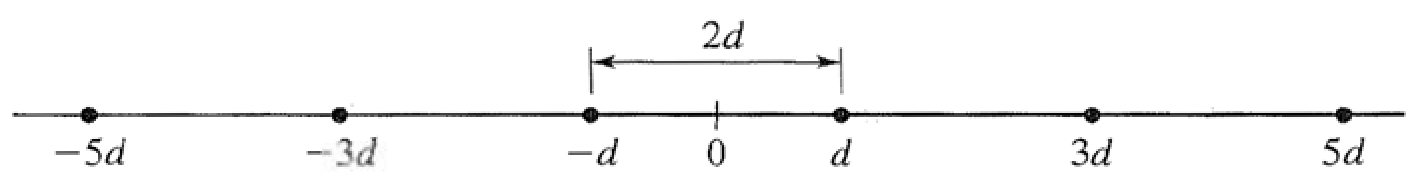


**Figure 8.29** Signal point constellation for $M$-ary PAM.

# $M$-ary Pulse Amplitude Modulation (5/5)

- **Example 8.4.1.** (Cont'd) The four signal waveforms are shown in Fig. 8.30

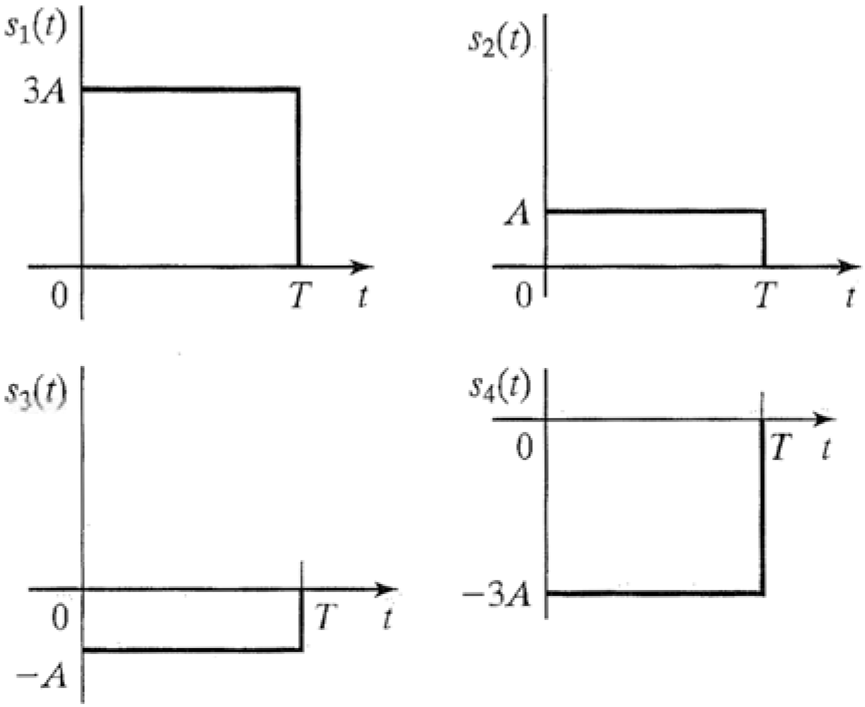


**Figure 8.30** $M = 4$ PAM signal waveforms.

# *M*-ary Orthogonal Signals (1/8)

- *M*-ary orthogonal signal waveforms at baseband can be constructed in a variety of ways. Fig. 8.31 illustrates two sets of $M=4$ orthogonal signal waveforms, which are represented as $s_i(t)$ and $s_i'(t)$, $i=1,\ldots,4$
- Fig. 8.31(a) and Fig. 8.31(b) both satisfy the orthogonality condition, namely,

$$\int_0^T s_i'(t)s_j'(t)dt = 0, \quad i \neq j$$

- The number of dimensions required to represent a set of *M* orthogonal waveforms is $N=M$. Hence, a set of *M* orthogonal signal waveforms can be represented geometrically by *M* orthogonal vectors in *M*-dimensional space

# M-ary Orthogonal Signals (2/8)

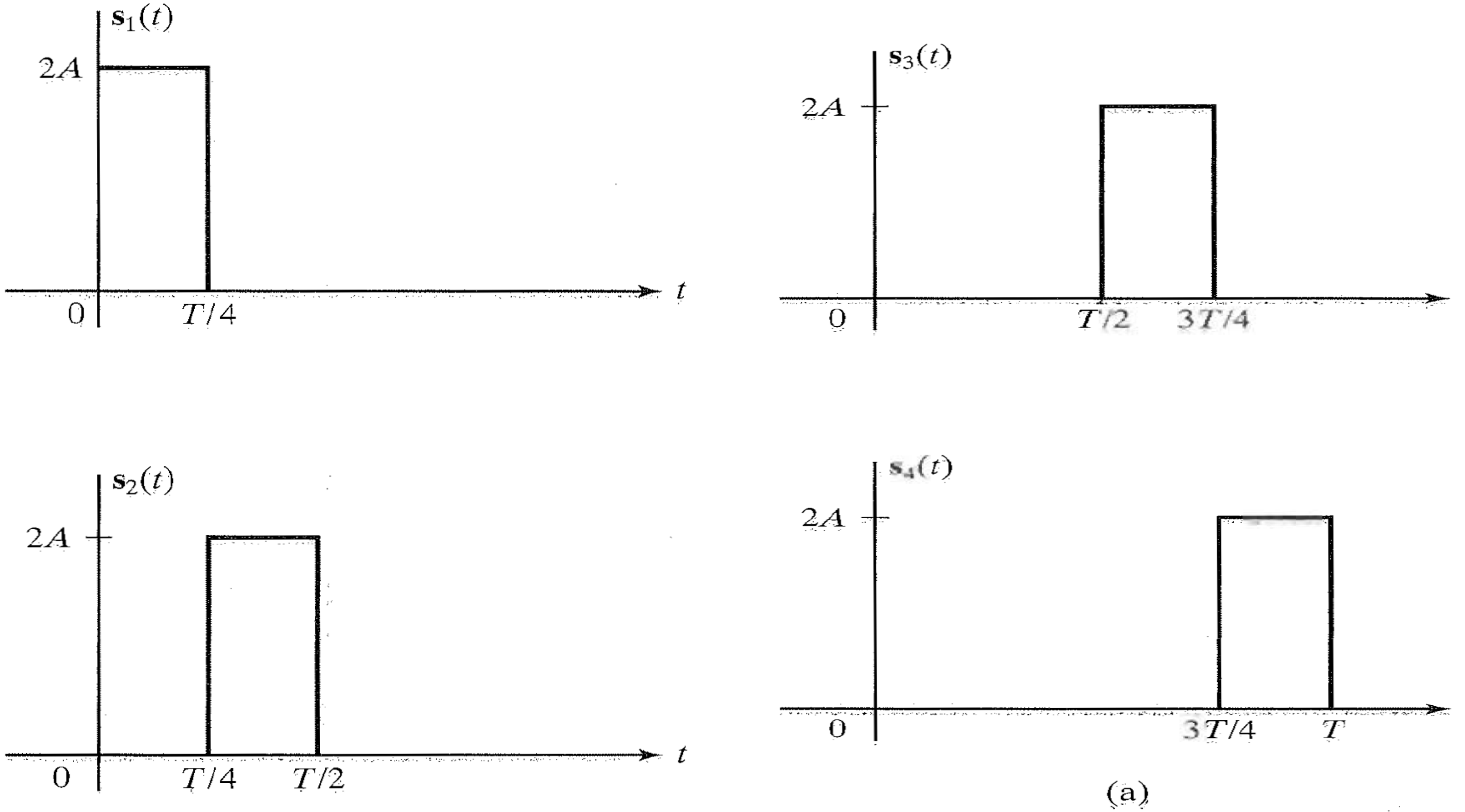


**Figure 8.31** Two sets of $M = 4$ orthogonal signal waveforms.

# $M$-ary Orthogonal Signals (3/8)

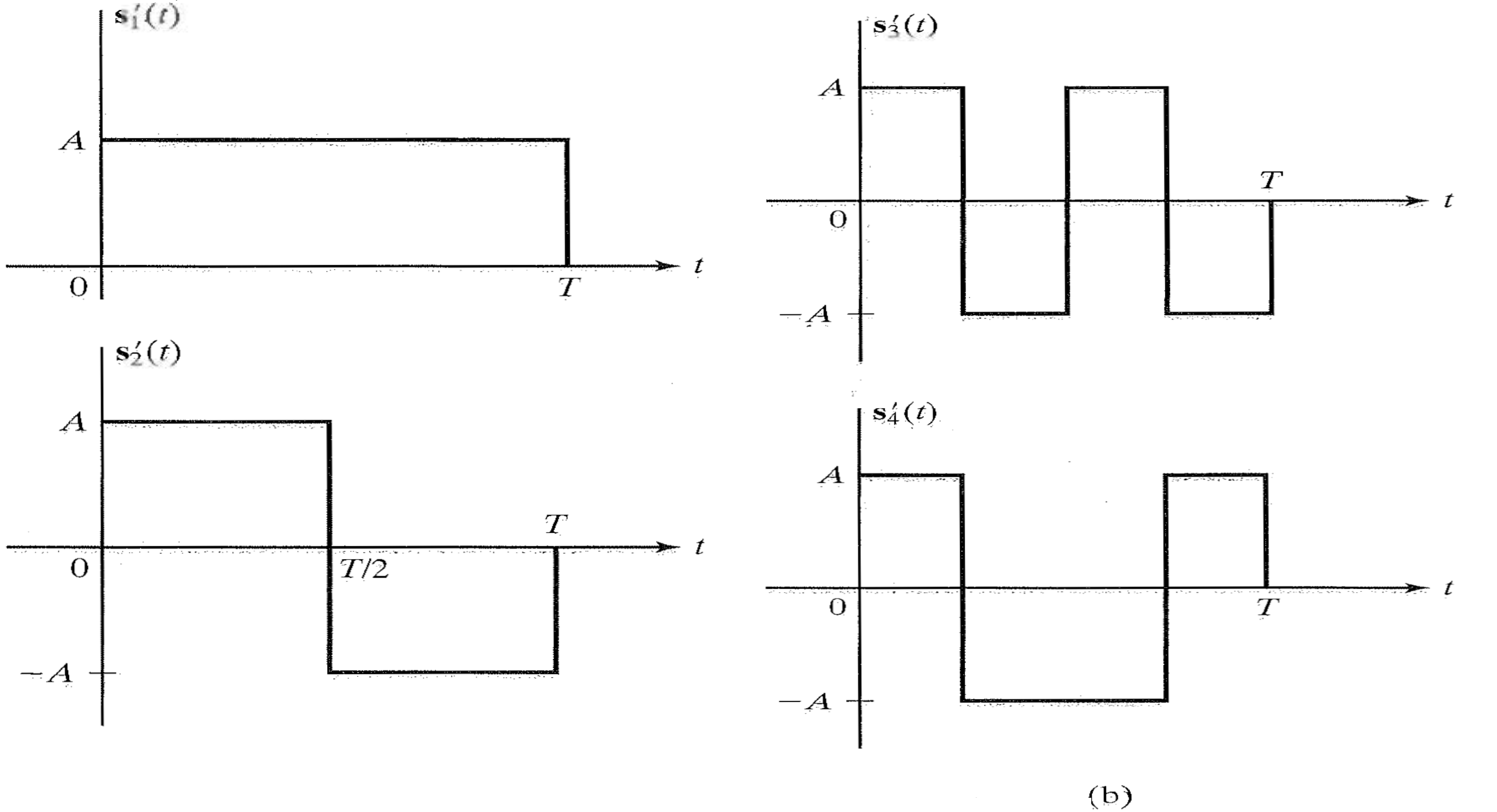


**Figure 8.31** Two sets of $M = 4$ orthogonal signal waveforms.

# $M$-ary Orthogonal Signals (4/8)

- Consider $M$-ary PPM signal waveforms expressed mathematically as

$$s_m(t) = \sqrt{\mathcal{E}_s}\,\psi_m(t),\ \ m = 1,2,\ldots,M\ ,$$

where $\psi_m(t)$ and $m=1,2,\ldots,M$ are a set of $M$ orthogonal basis waveforms. These waveform are defined as

$$\psi_m(t) = g_T\left(t - \tfrac{(m-1)T}{M}\right),\ \ \tfrac{(m-1)T}{M} \le t \le \tfrac{mT}{M}$$

in which $g_T(t)$ is a unit energy pulse, which is nonzero over the time interval $0 \le t \le \frac{T}{M}$

# $M$-ary Orthogonal Signals (5/8)

- Each signal waveform $s_m(t)$ has energy

$$\int_0^T s_m^{\ 2}(t)dt = \mathcal{E}_s \int_0^T \psi_m^{\ 2}(t)dt = \mathcal{E}_s \text{, all } m$$

  $\mathcal{E}_s$ denotes the energy of each of the signal waveforms representing $k$-bit symbols

- Note $\psi_m(t)$ has unit power

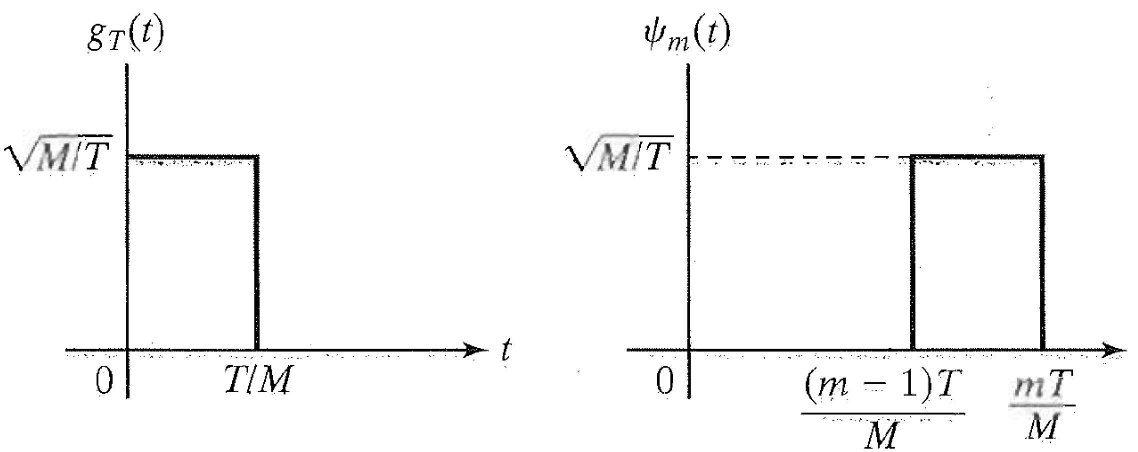


**Figure 8.32** Rectangular pulse $g_T(t)$ and basis function $\psi_m(t)$ for $M$-ary PPM signal waveforms.

# *M*-ary Orthogonal Signals (6/8)

- *M*-ary PPM signal waveforms are represented geometrically by the following *M*-dimensional vectors

$$\mathbf{s}_1 = (\sqrt{\mathcal{E}_s}, 0, 0, \cdots, 0)$$
$$\mathbf{s}_2 = (0, \sqrt{\mathcal{E}_s}, 0, \cdots, 0)$$
$$\vdots$$
$$\mathbf{s}_M = (0, 0, 0, \cdots, \sqrt{\mathcal{E}_s})$$

- These vectors are orthogonal, *i.e.*, $\mathbf{s}_i \bullet \mathbf{s}_j = 0$ when $i \neq j$
- The *M* signal vectors are mutually equidistant, *i.e.*,

$$d_{mn} = \| \mathbf{s}_m - \mathbf{s}_n \| = \sqrt{2\mathcal{E}_s}, \text{ for all } m \neq n$$

# $M$-ary Orthogonal Signals (7/8)

- **Example 8.4.2.** Determine the vectors in a geometric representation of the $M$=4 signal waveforms $s_i'(t)$ and $i$=1,2,3,4, that are shown in Fig. 8.31(b). Use the basis waveforms $\psi_m(t)$ that are shown in Fig. 8.32
- Note that the four orthogonal waveforms have equal energy, given by

$$\int_0^T [s_i'(t)]^2 dt = \mathcal{E}_s$$

- By computing the projection of each signal waveform on the four basis waveforms $\psi_m(t)$, *i.e.*,

$$\int_0^T s_i'(t)\psi_m(t)dt, \ \ m = 1,2,3,4,$$

we obtain the vector $\boldsymbol{s}_i'$

# $M$-ary Orthogonal Signals (8/8)

- **Example 8.4.2. (Cont'd)** We obtain

$$\begin{aligned}
\mathbf{s}_1' &= \left(\sqrt{\mathcal{E}_s/4},\ \sqrt{\mathcal{E}_s/4},\ \sqrt{\mathcal{E}_s/4},\ \sqrt{\mathcal{E}_s/4}\right)\\
\mathbf{s}_2' &= \left(\sqrt{\mathcal{E}_s/4},\ \sqrt{\mathcal{E}_s/4},\ -\sqrt{\mathcal{E}_s/4},\ -\sqrt{\mathcal{E}_s/4}\right)\\
\mathbf{s}_3' &= \left(\sqrt{\mathcal{E}_s/4},\ -\sqrt{\mathcal{E}_s/4},\ \sqrt{\mathcal{E}_s/4},\ -\sqrt{\mathcal{E}_s/4}\right)\\
\mathbf{s}_4' &= \left(\sqrt{\mathcal{E}_s/4},\ -\sqrt{\mathcal{E}_s/4},\ -\sqrt{\mathcal{E}_s/4},\ \sqrt{\mathcal{E}_s/4}\right)
\end{aligned}$$

- We observe that these four signal vectors are orthogonal, *i.e.*, $\mathbf{s}_i' \cdot \mathbf{s}_j' = 0$, for $i \neq j$
- Note that $\|\mathbf{s}_i'\|^2 = \mathcal{E}_s$, $i = 1,2,3,4$

# The Optimum Demodulator for *M*-ary Signals in AWGN (1/14)

- We assume that the digital communication system transmits digital information by using any of the *M*-ary signal waveforms described in the preceding sections
- Each of the $M=2^k$ symbols is associated with a corresponding baseband signal waveform from the set $\{s_m(t), m=1,2,\ldots,M\}$. Each signal waveform is transmitted within the symbol (signaling) interval *T*. We consider the transmission of information over the interval $0 \leqq t \leqq T$
- We wish to design a receiver that is optimum in the sense that it minimizes the probability of making an error

# The Optimum Demodulator for *M*-ary Signals in AWGN (2/14)

- The channel is assumed to corrupt the signal by the addition of white Gaussian noise. Thus, the received signal in the interval $0 \leqq t \leqq T$ may be expressed as

$$r(t) = s_m(t) + n(t),\ 0 \leqq t \leqq T$$

  where $n(t)$ denotes the sample function of the additive white Gaussian noise (AWGN) process with the power spectral density $S_n(f) = N_0/2$

- We subdivide the receiver into two parts: the signal demodulator and the detector. The demodulator converts the received waveform $r(t)$ into an $N$-dimensional vector $y = (y_1, y_2, \ldots, y_N)$, where $N$ is the dimension of the transmitted signal waveform

# The Optimum Demodulator for *M*-ary Signals in AWGN (3/14)

- The function of the detector is to decide which of the $M$ possible signal waveforms was transmitted based on observation of the vector $\boldsymbol{y}$
- We have shown that the $M$-ary signal waveform, each of which is $N$-dimensional, may be represented in general as

$$s_m(t) = \sum_{k=1}^{N} s_{mk}\psi_k(t),\ 0 \le t \le T,\ m = 1,2,\ldots,M,$$

where $\{s_{mk}\}$ are the coordinates of the signal vector

$$\boldsymbol{s}_m = (s_{m1}, s_{m2}, \ldots, s_{mN}),\ m = 1,2,\ldots,M,$$

and $\psi_k(t)$ and $k = 1,2,\ldots,N$ are $N$ orthonormal basis waveforms that span the $N$-dimensional signal space

# The Optimum Demodulator for *M*-ary Signals in AWGN (4/14)

- As a generalization of the demodulator for binary signals, we employ either a correlation-type demodulator or a matched-filter-type demodulator
- The correlator outputs at the end of the signal interval is

$$\begin{aligned} y_k &= \int_0^T r(t)\psi_k(t)dt \\ &= \int_0^T [s_m(t)+n(t)]\psi_k(t)dt \\ &= \int_0^T s_m(t)\psi_k(t)dt + \int_0^T n(t)\psi_k(t)dt \\ &= s_{mk} + n_k, \quad k = 1,2,\ldots, N \qquad (8.4.34) \end{aligned}$$

- Eq. (8.4.34) is equivalent to

$$\boldsymbol{y} = \boldsymbol{s}_m + \boldsymbol{n},$$

where $\boldsymbol{s}_m$ and $\boldsymbol{n}$ are vectors

# The Optimum Demodulator for *M*-ary Signals in AWGN (5/14)

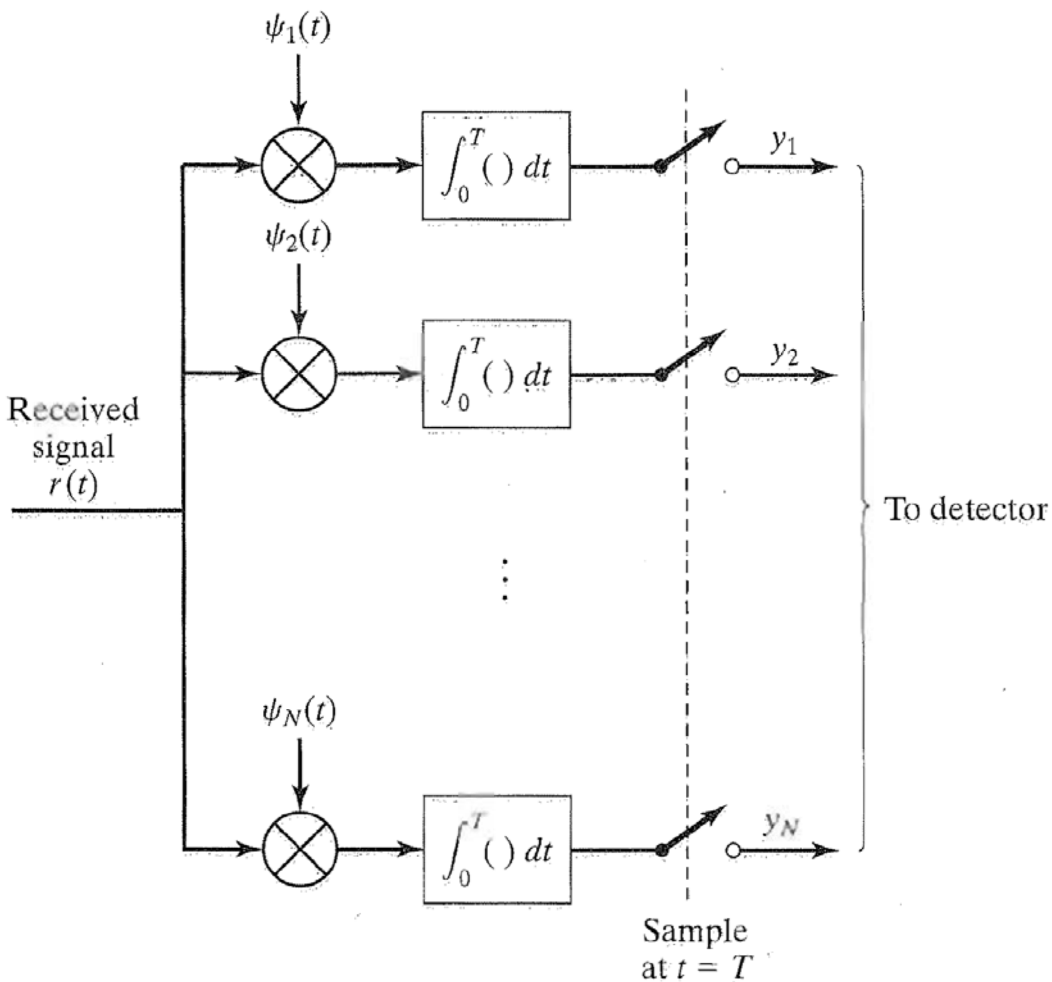


**Figure 8.39** Correlation-type demodulator.

# The Optimum Demodulator for *M*-ary Signals in AWGN (6/14)

- We can express the received signal $r(t)$ in the interval $0 \leqq t \leqq T$ as

$$r(t) = \sum_{k=1}^{N} s_{mk}\psi_k(t) + \sum_{k=1}^{N} n_k\psi_k(t) + n'(t)$$
$$= \sum_{k=1}^{N} y_k\psi_k(t) + n(t)$$

- The term $n'(t)$, defined as

$$n'(t) = n(t) - \sum_{k=1}^{N} n_k\psi_k(t),$$

  is a zero-mean Gaussian noise process that represents the difference between the original noise process $n(t)$ and the part that corresponds to the projection of $n(t)$ onto the basis function $\{\psi_k(t)\}$

# The Optimum Demodulator for *M*-ary Signals in AWGN (7/14)

- Note that $y_k = s_{mk} + n_k$, $k = 1, 2, \ldots, N$. Since the signal $\{s_m(t)\}$ are deterministic, the signal components $\{s_{mk}\}$ are deterministic. The noise components $\{n_k\}$ are Gaussian distributed
- The mean values of $\{n_k\}$ are

$$E[n_k] = \int_0^T E[n(t)]\psi_k(t)dt = 0.$$

Their covariance are

$$\begin{aligned} E[n_k n_m] &= \int_0^T \int_0^T E[n(t)n(\tau)]\psi_k(t)\psi_m(\tau)dtd\tau \\ &= \int_0^T \int_0^T \tfrac{N_0}{2}\delta(t-\tau)\psi_k(t)\psi_m(\tau)dtd\tau \\ &= \tfrac{N_0}{2}\int_0^T \psi_k(t)\psi_m(t)dt \\ &= \tfrac{N_0}{2}\delta_{mk}, \end{aligned}$$

where $\delta_{mk}$ is the Kronecker delta. Note $\delta_{mk}=1$ when $m=k$ and will otherwise be zero

# The Optimum Demodulator for *M*-ary Signals in AWGN (8/14)

- The $N$ noise components $\{n_k\}$ are zero-mean uncorrelated Gaussian random variables with a common variance $\sigma_n^2 = N_0/2$, and it follows that

$$f(\mathbf{n}) = \prod_{i=1}^{N} f(n_i) = \frac{1}{(\pi N_0)^{N/2}} e^{-\sum_{i=1}^{N} \frac{n_i^2}{N_0}}$$

- The correlator output $\{y_k\}$ conditioned on the $m$th signal being transmitted are Gaussian random variables with mean

$$E[y_k] = E[s_{mk} + n_k] = s_{mk}$$

and equal variance

$$\sigma_y^2 = \sigma_n^2 = N_0/2.$$

Since the noise components $\{n_k\}$ are uncorrelated Gaussian random variables they are also statistically independent

# The Optimum Demodulator for *M*-ary Signals in AWGN (9/14)

- The conditional probability density functions (PDF's) of the variables $(y_1, y_2, \ldots, y_N) = \mathbf{y}$ are simply

$$f(\mathbf{y} \mid \mathbf{s}_m) = \prod_{k=1}^{N} f(y_k \mid s_{mk}), \quad m = 1,2,\ldots,M$$

where

$$f(y_k \mid s_{mk}) = \frac{1}{\sqrt{\pi N_0}} e^{-(y_k - s_{mk})^2 / N_0}, \quad k = 1,2,\ldots,N$$

- We obtain the joint conditional PDF's as

$$f(\mathbf{y} \mid \mathbf{s}_m) = \frac{1}{(\pi N_0)^{N/2}} \exp\left[-\sum_{k=1}^{N} (y_k - s_{mk})^2 / N_0\right]$$
$$= \frac{1}{(\pi N_0)^{N/2}} \exp\left[-\left\| \mathbf{y} - \mathbf{s}_m \right\|^2 / N_0\right], \quad m = 1,2,\ldots,M$$

- We can show that no additional relevant information can be extracted from the remaining noise $n'(t)$, *i.e.*, $E[n'(t)y_k] = 0$

# The Optimum Demodulator for *M*-ary Signals in AWGN (10/14)

$$
\begin{aligned}
E[n^{'}(t)y_k] &= E[n^{'}(t)(s_{mk}+n_k)] \\
&= E[n^{'}(t)s_{mk}] + E[n'(t)n_k] \\
&= E[n^{'}(t)]s_{mk} + E[n'(t)n_k] \\
&= E[n'(t)n_k] \\
&= E\left[\left(n(t) - \sum_{j=1}^{N} n_j \psi_j(t)\right) n_k\right] \\
&= E\left[n(t)n_k - \sum_{j=1}^{N} n_j n_k \psi_j(t)\right] \\
&= \int_0^T E[n(t)n(\tau)]\psi_k(t)dt - \sum_{j=1}^{N} E(n_j n_k)\psi_j(t) \\
&= \tfrac{N_0}{2}\psi_k(t) - \tfrac{N_0}{2}\psi_k(t) \\
&= 0
\end{aligned}
$$

# The Optimum Demodulator for *M*-ary Signals in AWGN (11/14)

- Since $n'(t)$ and $\{y_k\}$ are Gaussian and uncorrelated, they are also statistically independent. All the relevant information is contained in the correlator output $\{y_k\}$
- **Example 8.4.5.** Consider an *M*-ary PAM signal set given by $s_m(t) = s_m \psi(t),\ 0 \le t \le T,\ \ m = 1,2,\ldots,M$. The basis function is defined by $\psi(t) = \sqrt{1/T},\ 0 \le t \le T$. The additive noise is a zero-mean white Gaussian noise process with spectral density $N_0/2$. Determine the PDF of the received signal at the output of the demodulator and sketch the PDFs for the case $M$=4.
- The received signal is expressed as

$$r(t) = s_m \psi(t) + n(t).$$

# The Optimum Demodulator for *M*-ary Signals in AWGN (12/14)

- **Example 8.4.5.** (Cont'd) The output of the demodulator is

$$y(T) = \int_0^T r(t)\psi(t)dt = \int_0^T [s_m\psi(t) + n(t)]\psi(t)dt$$

$$= s_m + n,$$

where $n$ is a zero-mean Gaussian random variable with variance $\sigma_n^2 = N_0/2$. Therefore, the PDF of $y \equiv y(T)$ is

$$f(y \mid s_m) = \frac{1}{\sqrt{\pi N_0}} e^{-(y-s_m)^2/N_0}, \quad m = 1,2,\ldots,M$$

and $s_m = (2m-1-M)d$. The PDF's for $M$=4 PAM are shown below

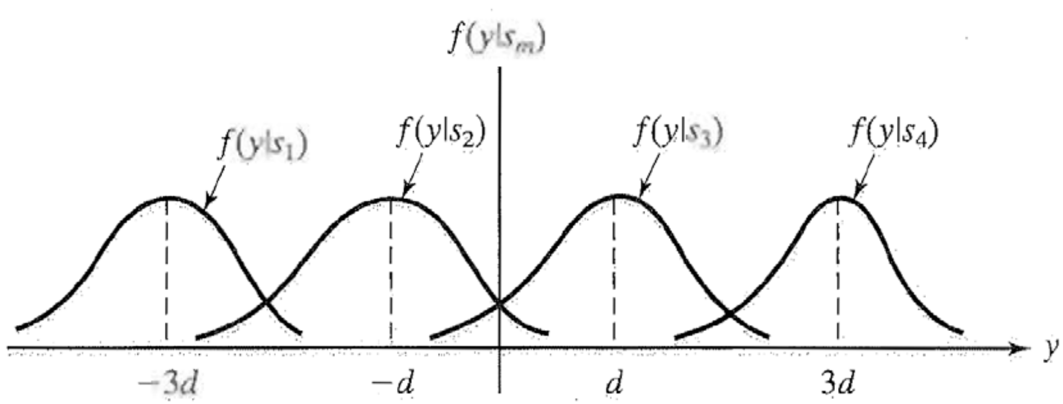


**Figure 8.41** PDF's for $M = 4$ received PAM signals in additive white Gaussian noise.

# The Optimum Demodulator for *M*-ary Signals in AWGN (13/14)

- **Example 8.4.6.** Consider the $M=4$ orthogonal PPM signal waveforms shown in Fig. 8.31(a), where the signal constellation points $\{\boldsymbol{s}_m\}$ are given by

$$\boldsymbol{s}_1 = (\sqrt{\mathcal{E}_s}, 0, 0, 0);$$
$$\boldsymbol{s}_2 = (0, \sqrt{\mathcal{E}_s}, 0, 0);$$
$$\vdots$$
$$\boldsymbol{s}_4 = (0, 0, 0, \sqrt{\mathcal{E}_s}).$$

The additive noise is zero-mean white Gaussian noise process with a spectral density $N_0/2$. Determine the PDF of the received signal vector $\boldsymbol{y}$ at the output of the demodulator, assuming that the signal $s_1(t)$ was transmitted

# The Optimum Demodulator for *M*-ary Signals in AWGN (14/14)

- **Example 8.4.6.** (Cont'd) The received signal vector is

$$\boldsymbol{y} = \boldsymbol{s}_1 + \boldsymbol{n}$$
$$= (\sqrt{\mathcal{E}_s} + n_1, n_2, n_3, n_4),$$

  where the noise components $n_1, n_2, n_3, n_4$ are mutually statistically independent, zero-mean Gaussian random variables with identical variance $\sigma_n^2 = N_0/2$

- The joint PDF of the vector components $y_1, y_2, y_3, y_4$ is

$$f(y_1, y_2, y_3, y_4 \mid \boldsymbol{s}_1) = \frac{1}{(\pi N_0)^2} e^{-\left[(y_1 - \sqrt{\mathcal{E}_s})^2 + y_2^2 + y_3^2 + y_4^2\right]/N_0}$$