Chapter 8 Digital Modulation in an Additive White Gaussian Noise Baseband Channel (III)

Correlation-Type Demodulator (1/10)

- Let us consider the two orthogonal signals given in Fig. 8.10, where $\psi_1(t)$ and $\psi_2(t)$ are the orthogonal basis functions shown in Fig. 8.11
- $\boldsymbol{s}_1=(s_{11},0)=(\sqrt{\mathcal{E}_b},0)$ and $\boldsymbol{s}_2=(0,s_{22})=(0,\sqrt{\mathcal{E}_b})$

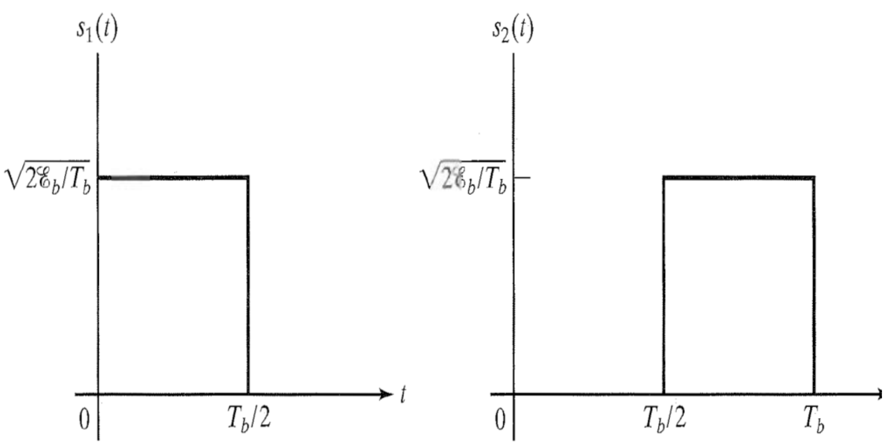


Figure 8.10 Signal pulses in binary PPM (orthogonal signals).

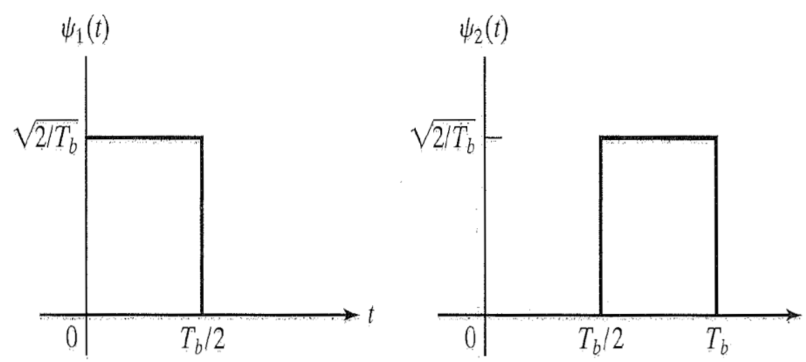


Figure 8.11 Two orthogonal basis functions for binary PPM signals.

Correlation-Type Demodulator (2/10)

- In the presence of AWGN, the received signal has the form

$$r(t) = s_m(t) + n(t),\ 0 \leqq t \leqq T_b,\ m = 1, 2$$

- The received signal $r(t)$ is cross-correlated with each of the two basis signal waveforms $\psi_1(t)$ and $\psi_2(t)$, as shown in Fig. 8.20

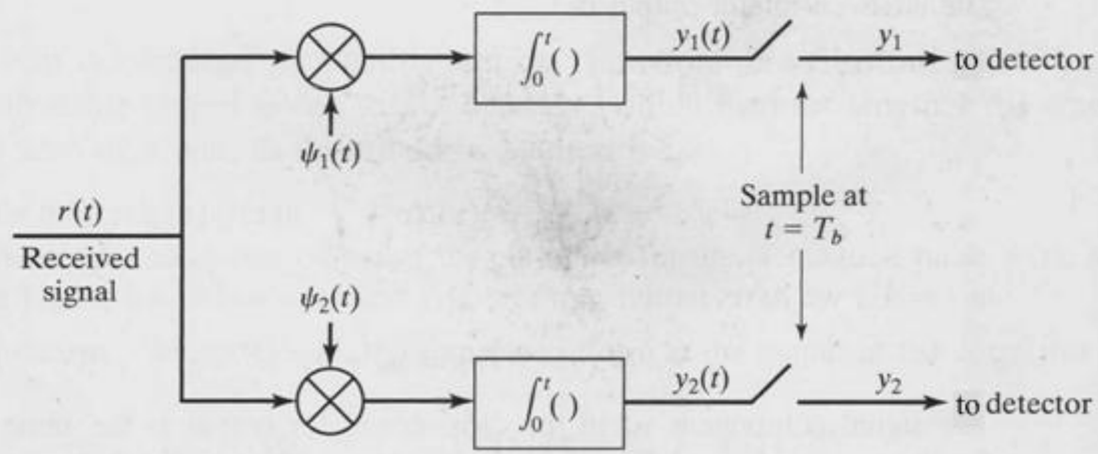


Figure 8.20 Correlation-type demodulator for binary orthogonal signals.

Correlation-Type Demodulator (3/10)

- The correlator output waveforms are

$$y_m(t) = \int_0^t r(\tau)\psi_m(\tau)d\tau, \quad m = 1,2,$$

which, when sampled at $t = T_b$, result in the outputs

$$y_m = y_m(T_b) = \int_0^{T_b} r(\tau)\psi_m(\tau)d\tau, \quad m = 1,2$$

- Suppose the transmitted signal is $s_1(t) = s_{11}\psi_1(t)$ signal, so that $r(t) = s_{11}\psi_1(t) + n(t)$. The output of the first correlator is

$$y_1 = \int_0^{T_b} [s_{11}\psi_1(\tau) + n(\tau)]\psi_1(\tau)d\tau$$
$$= s_{11} + n_1 = \sqrt{\mathcal{E}_b} + n_1,$$

where s_{11} is the signal component and n_1 is the noise component

$$n_1 = \int_0^{T_b} n(\tau)\psi_1(\tau)d\tau$$

Correlation-Type Demodulator (4/10)

- The output of the second correlator is

$$\begin{aligned} y_2 &= \int_0^{T_b} [s_{11}\psi_1(\tau) + n(\tau)]\psi_2(\tau)d\tau \\ &= s_{11}\int_0^{T_b} \psi_1(\tau)\psi_2(\tau)d\tau + \int_0^{T_b} n(\tau)\psi_2(\tau)d\tau \\ &= \int_0^{T_b} n(\tau)\psi_2(\tau)d\tau = n_2 \end{aligned}$$

- The output of the second correlator only includes the noise component n_2, because $\psi_1(t)$ and $\psi_2(t)$ are orthogonal
- The received signal vector is

$$\begin{aligned} y &= (y_1, y_2) \\ &= (\sqrt{\mathcal{E}_b} + n_1, n_2) \end{aligned}$$

Correlation-Type Demodulator (5/10)

- It is easy to verify that when the signal $s_2(t)=s_{22}\psi_2(t)$ is transmitted, the outputs of the two correlators are $y_1=n_1$ and $y_2=s_{22}+n_2=\sqrt{\mathcal{E}_b}+n_2$. The received signal vector is

$$\begin{aligned} y &= (y_1, y_2) \\ &= (n_1, \sqrt{\mathcal{E}_b}+n_2) \end{aligned}$$

- The vector y at the output of the cross-correlators is fed to the detector, which decides whether the received signal vector corresponds to the transmission of a one or a zero
- Since $n(t)$ is a sample function of a white Gaussian noise process, the noise terms n_1 and n_2 are zero-mean Gaussian random variables with variance $\sigma_n^2=N_0/2$

Correlation-Type Demodulator (6/10)

- The correlation between n_1 and n_2 is

$$
\begin{aligned}
E(n_1 n_2) &= \int_0^{T_b}\int_0^{T_b} E[n(t)n(\tau)]\psi_1(t)\psi_2(\tau)\,dt\,d\tau \\
&= \int_0^{T_b}\int_0^{T_b} \tfrac{N_0}{2}\delta(t-\tau)\psi_1(t)\psi_2(\tau)\,dt\,d\tau \\
&= \tfrac{N_0}{2}\int_0^{T_b}\psi_1(t)\psi_2(t)\,dt = 0
\end{aligned}
$$

- n_1 and n_2 are uncorrelated; since they are Gaussian, n_1 and n_2 are statistically independent
- When the transmitted signal is $s_1(t)$, the conditional joint probability density function of the correlator output components (y_1, y_2) is

$$
f(y_1, y_2 \mid \boldsymbol{s}_1) = \left(\tfrac{1}{\sqrt{\pi N_0}}\right)^2 e^{-(y_1-\sqrt{\mathcal{E}b})^2/N_0} e^{-y_2{}^2/N_0}
$$

Correlation-Type Demodulator (7/10)

- When $s_2(t)$ is transmitted, the conditional joint probability density function of the correlator components (y_1, y_2) is

$$f(y_1, y_2 \mid \boldsymbol{s}_2) = \left(\frac{1}{\sqrt{\pi N_0}}\right)^2 e^{-y_1^2/N_0} e^{-(y_2 - \sqrt{\mathcal{E}_b})^2/N_0}$$

- Since the noise components n_1 and n_2 are statistically independent, we observe that the joint probability density functions of (y_1, y_2) factor into a product of marginal probability density function, *i.e.*,

$$f(y_1, y_2 \mid \boldsymbol{s}_m) = f(y_1 \mid \boldsymbol{s}_m) f(y_2 \mid \boldsymbol{s}_m), \quad m = 1, 2$$

Correlation-Type Demodulator (8/10)

- Fig. 8.21 shows the probability density functions $f(y_1 \mid \boldsymbol{s}_m)$ and $f(y_2 \mid \boldsymbol{s}_m)$ when $s_1(t)$ is transmitted

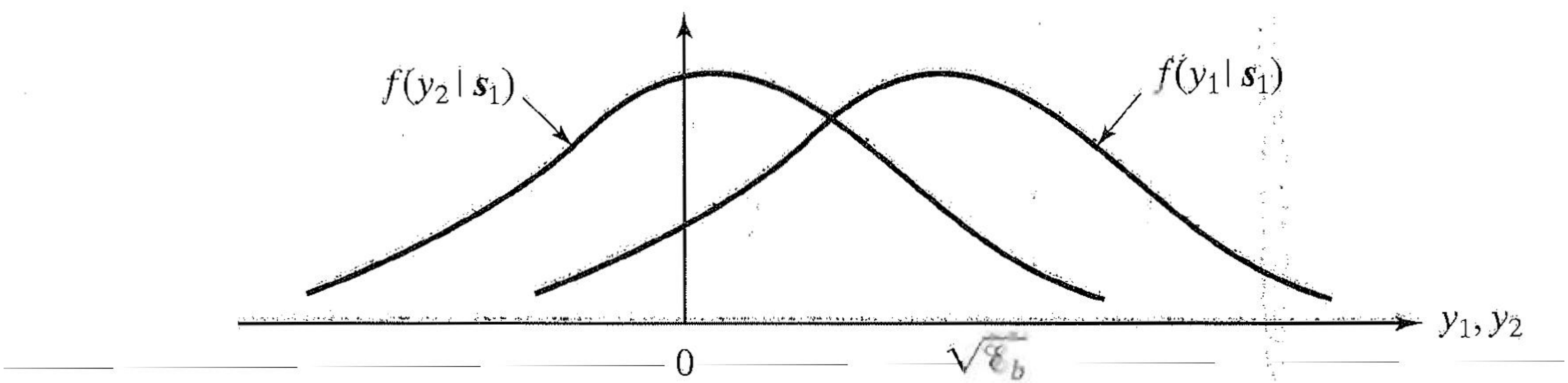


Figure 8.21 The conditional probability density functions of the outputs (y_1, y_2) from the cross-correlators of two orthogonal signals.

Correlation-Type Demodulator (9/10)

- **Example 8.3.3.** Assuming that the transmitted waveform is $s_2(t) = \sqrt{\mathcal{E}_b}\psi_2(t)$, in the absence of additive noise, sketch the output waveforms of the two correlators shown in Fig. 8.20. The other signal waveform is $s_1(t) = \sqrt{\mathcal{E}_b}\psi_1(t)$, where $\psi_1(t)$ and $\psi_2(t)$ are the basis functions shown in Fig. 8.11
- When $s_2(t)$ is transmitted in the absence of noise, the output of the two correlators are

$$y_1(t) = \int_0^t s_2(\tau)\psi_1(\tau)d\tau = \sqrt{\mathcal{E}_b}\int_0^t \psi_2(\tau)\psi_1(\tau)d\tau$$
$$y_2(t) = \int_0^t s_2(\tau)\psi_2(\tau)d\tau = \sqrt{\mathcal{E}_b}\int_0^t \psi_2{}^2(\tau)d\tau$$

Correlation-Type Demodulator (10/10)

- **Example 8.3.3. (Cont'd)** Note that the noise-free output of the first correlator is zero for $0 \leqq t \leqq T_b$ because $\psi_1(t)$ and $\psi_2(t)$ are nonoverlapping orthogonal waveforms

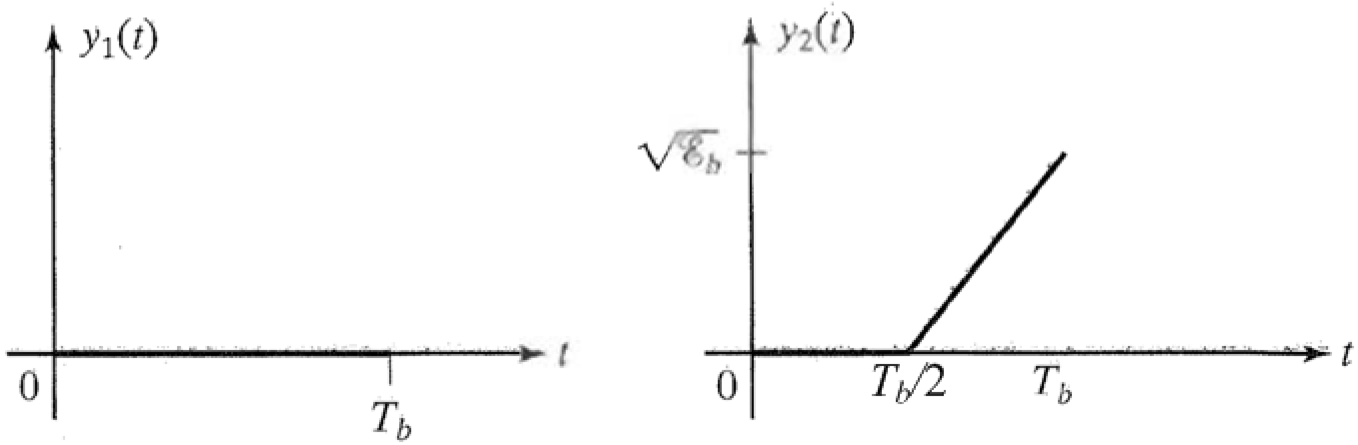


Figure 8.22 Correlator output signal waveforms when $s_2(t)$ is the transmitted signal.

Matched-Filter-Type Demodulator (1/6)

- We may use a filter-type demodulator and we first consider binary antipodal signals
- In the case of binary antipodal signals, the received signal is

$$r(t) = s_m \psi(t) + n(t),\ 0 \le t \le T_b,\ \ m = 1,2,$$

where $\psi(t)$ is a unit energy rectangular pulse

- Suppose we pass the received signal $r(t)$ through a linear, time-invariant filter with impulse response

$$h(t) = \psi(T_b - t),\ 0 \le t \le T_b.$$

The filter output is

$$y(t) = \int_0^t r(\tau) h(t-\tau) d\tau$$

Matched-Filter-Type Demodulator (2/6)

- If we sample the output of the filter at $t=T_b$, we obtain

$$y(T_b) = \int_0^{T_b} r(\tau)h(T_b - \tau)d\tau.$$

But $h(T_b - t) = \psi(t)$. Therefore,

$$\begin{aligned} y(T_b) &= \int_0^{T_b} [s_m\psi(\tau) + n(\tau)]\psi(\tau)d\tau. \\ &= s_m \int_0^{T_b} \psi^2(\tau)d\tau + \int_0^{T_b} n(\tau)\psi(\tau)d\tau \\ &= s_m + n \end{aligned}$$

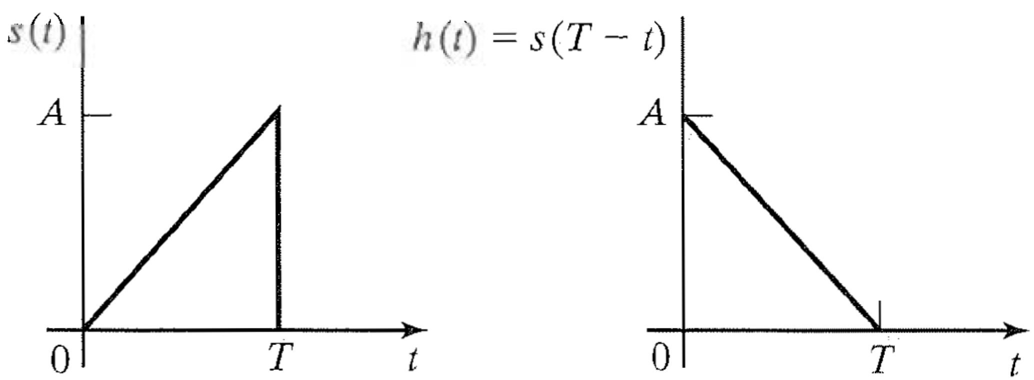
- The output of the filter at $t=T_b$ is exactly the same as the output obtained with a cross-correlator
- A filter whose impulse response $h(t)=s(T_b-t)$, where $s(t)$ is assumed to be confined to the time interval $0 \leqq t \leqq T_b$, is called the *matched filter* to the signal $s(t)$

Matched-Filter-Type Demodulator (3/6)

- An example of a signal and its matched filter are shown in Fig. 8.23. The response of $h(t)=s(T\text{-}t)$ to the signal $s(t)$ is

$$y(t) = \int_0^t s(\tau)s(T-t+\tau)d\tau$$

- $y(t)$ is basically the time-autocorrelation function of the signal $s(t)$



(a) Signal $s(t)$

(b) Impulse response of filter matched to $s(t)$

Figure 8.23 Signal $s(t)$ and filter matched to $s(t)$.

Matched-Filter-Type Demodulator (4/6)

- When we use $s(t)$ and $h(t)$ defined in Fig. 8.23, $y(t)$ for the triangular signal pulse is shown in Fig. 8.24
- The autocorrelation function $y(t)$ is an even function of t, which attains a peak at $t=T$. The peak value $y(T)$ is equal to the energy of the signal $s(t)$

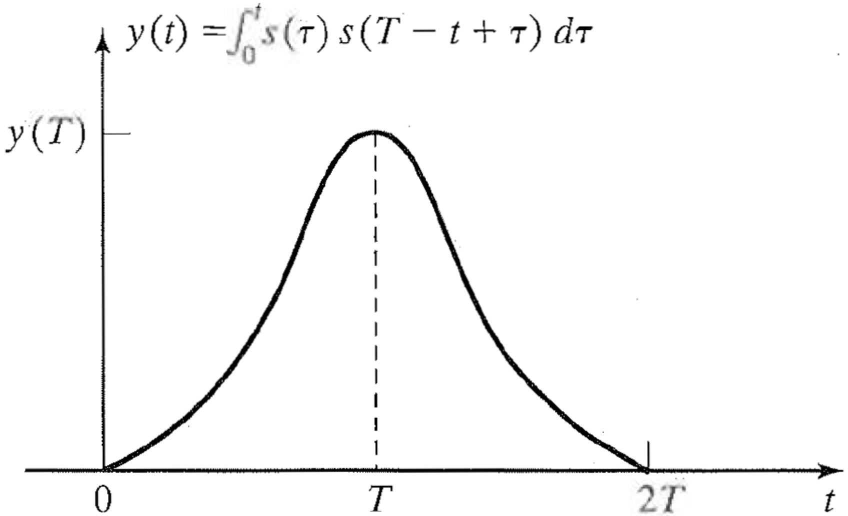


Figure 8.24 The matched filter output is the autocorrelation function of $s(t)$.

Matched-Filter-Type Demodulator (5/6)

- We next consider binary orthogonal signals. Orthogonal signals are two-dimensional signals; hence, two linear time-invariant filters should be employed
- Consider the received signal

$$r(t) = s_m(t) + n(t),\ 0 \le t \le T_b,\ \ m = 1,2,$$

where $s_m(t)$ and $m=1, 2$ are the two orthogonal waveforms

- The impulse responses of the two filters matched to $\psi_1(t)$ and $\psi_2(t)$ are defined as

$$h_1(t) = \psi_1(T_b - t),\ 0 \le t \le T_b$$

and

$$h_2(t) = \psi_2(T_b - t),\ 0 \le t \le T_b$$

Matched-Filter-Type Demodulator (6/6)

- When the received signal $r(t)$ is passed through the two filters, their outputs are

$$y_m(t) = \int_0^t r(\tau) h_m(t-\tau)\, d\tau, \quad m = 1,2.$$

If we sample the outputs of these filters at $t=T_b$, we obtain

$$\begin{aligned} y_m = y_m(T_b) &= \int_0^{T_b} r(\tau) h_m(T_b-\tau)\, d\tau \\ &= \int_0^{T_b} r(\tau)\psi_m(\tau)\, d\tau, \quad m = 1,2 \qquad (8.3.29) \end{aligned}$$

- (8.3.29) is the same as the outputs obtained from the cross-correlators. The correlation-type demodulator and the matched filter-type demodulator yield identical outputs at $t=T_b$

Properties of the Matched Filter (1/4)

- The most important property of a matched filter is stated as follows: If a signal is corrupted by AWGN, the filter with the impulse response matched to $s(t)$ maximizes the output signal-to-noise ratio (SNR)
- Assume that the received signal $r(t)$ consists of the signal $s(t)$ and AWGN $n(t)$, which has zero mean and a power-spectral density $S_n(f)=N_0/2$ W/Hz. Suppose the signal $r(t)$ is passed through a filter with the impulse response $h(t)$, $0 \leqq t \leqq T$
- The filter response to the signal and noise components is

$$\begin{aligned} y(t) &= \int_0^t r(\tau)h(t-\tau)d\tau \\ &= \int_0^t s(\tau)h(t-\tau)d\tau + \int_0^t n(\tau)h(t-\tau)d\tau \end{aligned}$$

Properties of the Matched Filter (2/4)

- At the sampling instant $t=T$, the signal and noise components are

$$y(T)=\int_0^T s(\tau)h(T-\tau)d\tau+\int_0^T n(\tau)h(T-\tau)d\tau$$
$$=y_s(T)+y_n(T),$$

- The problem is to select the filter impulse response that maximizes the output SNR, defined as

$$\left(\frac{S}{N}\right)_o=\frac{y_s^{\,2}(T)}{E[y_n^{\,2}(T)]} \qquad (8.3.32)$$

- Let us evaluate $E[y_n^{\,2}(T)]$. We have

$$E[y_n^{\,2}(T)]=\int_0^T\int_0^T E[n(\tau)n(t)]h(T-\tau)h(T-t)d\tau dt$$
$$=\frac{N_0}{2}\int_0^T\int_0^T \delta(t-\tau)h(T-\tau)h(T-t)d\tau dt \;=\frac{N_0}{2}\int_0^T h^2(T-\tau)dt$$

Properties of the Matched Filter (3/4)

- Substituting for $y_s(T)$ and $E[y_n^{\,2}(T)]$ in (8.3.32), we obtain the expression for the output SNR as

$$\left(\frac{S}{N}\right)_o = \frac{\left[\int_0^T s(\tau)h(T-\tau)d\tau\right]^2}{\frac{N_0}{2}\int_0^T h^2(T-t)dt} = \frac{\left[\int_0^T h(\tau)s(T-\tau)d\tau\right]^2}{\frac{N_0}{2}\int_0^T h^2(T-t)dt} \qquad (8.3.34)$$

- The denominator of the SNR depends on the energy in $h(t)$, the maximum output SNR over $h(t)$ is obtained by maximizing the numerator of $(S/N)_o$ subject to the constraint that the denominator is held constant
- The maximization of the numerator is performed by using Cauchy-Schwartz inequality,

$$\left[\int_{-\infty}^{\infty} g_1(t)g_2(t)dt\right]^2 \le \int_{-\infty}^{\infty} g_1^{\,2}(t)dt\int_{-\infty}^{\infty} g_2^{\,2}(t)dt$$

Properties of the Matched Filter (4/4)

- Equality holds when $g_1(t)=Cg_2(t)$ for any arbitrary constant C
- If we set $g_1(t)=h(t)$ and $g_2(t)=s(T\text{-}t)$, it is clear that the $(S/N)_o$ is maximized when $h(t)=Cs(T\text{-}t)$, *i.e.*, $h(t)$ is matched to the signal $s(t)$
- The output (maximum) SNR obtained with the matched filter is

$$\left(\frac{S}{N}\right)_o = \frac{2}{N_0}\int_0^T s^2(t)dt = \frac{2\mathcal{E}_s}{N_0} \qquad (8.3.36)$$

- $\mathcal{E}_s$ is the energy of the signal $s(t)$. Here, $\mathcal{E}_s$ is equivalent to bit energy $\mathcal{E}_b$. The output SNR depends on the energy of the waveform $s(t)$, but not on the detailed characteristics of $s(t)$

Frequency Domain Interpretation of the Matched Filter (1/8)

- Since $h(t)=s(T-t)$, the Fourier transform of this relationship is

$$\begin{aligned} H(f) &= \int_0^T s(T-t)e^{-j2\pi ft}dt \\ &= \left[\int_0^T s(\tau)e^{j2\pi f\tau}d\tau\right]e^{-j2\pi fT} \\ &= S^*(f)e^{-j2\pi fT}. \end{aligned}$$

  Note $s(t)$ is a real signal.

- The matched filter has a frequency response that is the complex conjugate of the transmitted signal spectrum multiplied by the phase factor $e^{-j2\pi fT}$
- In other words, $|H(f)|=|S(f)|$. The phase of $H(f)$ is the negative of the phase of $S(f)$, shifted by a linear function of T

Frequency Domain Interpretation of the Matched Filter (2/8)

- If the signal $s(t)$, with spectrum $S(f)$, is passed through the matched filter, the filter output has a spectrum $Y(f) = |S(f)|^2 e^{-j2\pi fT}$. Hence, the output waveform is

$$\begin{aligned} y_s(t) &= \int_{-\infty}^{\infty} Y(f) e^{j2\pi ft} df \\ &= \int_{-\infty}^{\infty} |S(f)|^2 e^{-j2\pi fT} e^{j2\pi ft} df \end{aligned}$$

- By sampling the output of the matched filter at $t=T$, we obtain

$$y_s(T) = \int_{-\infty}^{\infty} |S(f)|^2 df = \int_0^T s(t)^2 dt = \mathcal{E}_b,$$

where the last step follows from Parseval's relation

Frequency Domain Interpretation of the Matched Filter (3/8)

- The noise of the output of the matched filter has a power spectral density

$$S_0(f) = |H(f)|^2 N_0/2$$

- The total noise power at the output of the matched filter is

$$\begin{aligned} P_n &= \int_{-\infty}^{\infty} S_0(f)df \\ &= \int_{-\infty}^{\infty} \frac{N_0}{2} |H(f)|^2 df \\ &= \frac{N_0}{2} \int_{-\infty}^{\infty} |S(f)|^2 df \\ &= \frac{N_0}{2} \mathcal{E}_b \end{aligned}$$

- The signal power is

$$P_s = y_s^{\,2}(T)$$

Frequency Domain Interpretation of the Matched Filter (4/8)

- The output SNR is the ratio of the signal power to the noise power. Hence,

$$\left(\frac{S}{N}\right)_o = \frac{P_s}{P_n} = \frac{2\mathcal{E}_b}{N_0}$$

- The SNR agrees with (8.3.36). This means the matched filter demodulator is the optimal demodulator in an AWGN channel no matter what kind of binary signaling is used (both antipodal and orthogonal signaling are OK!)

Frequency Domain Interpretation of the Matched Filter (5/8)

- **Example 8.3.5.** Consider the binary orthogonal PPM signals, which are shown in Fig. 8.10, for transmitting information over an AWGN channel. Determine the impulse response of the matched filter demodulators and the output waveforms of the matched filter demodulators when the transmitted signal is $s_1(t)$

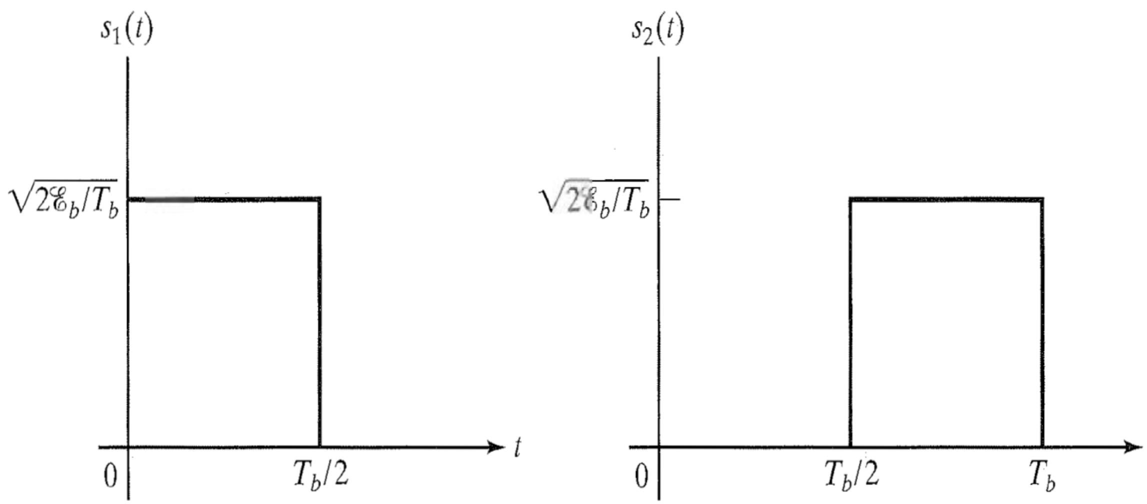


Figure 8.10 Signal pulses in binary PPM (orthogonal signals).

Frequency Domain Interpretation of the Matched Filter (6/8)

- **Example 8.3.5. (Cont'd)** We choose $\psi_1(t)$ and $\psi_2(t)$ as shown in Fig. 8.25(a). The impulse responses of the two matched filters are illustrated in Fig. 8.25(b).
- If $s_1(t)$ is transmitted, the noise-free responses of the two matched filters are shown in Fig. 8.25(c). Since $y_1(t)$ and $y_2(t)$ are sampled at $t=T_b$, the received vector is

$$y=(y_1,y_2)=(\sqrt{\mathcal{E}_b}+n_1,n_2),$$

where $n_1=y_{1n}(T_b)$ and $n_2=y_{2n}(T_b)$ are the noise components given by

$$y_{kn}(T_b)=\int_0^{T_b} n(t)\psi_k(t)dt, \qquad k=1,2$$

Frequency Domain Interpretation of the Matched Filter (7/8)

- Clearly, $E[n_k]=E[y_{kn}(T_b)]=0$. Their variance is

$$\sigma_n^{\ 2} = E[y_{kn}^{\ 2}(T_b)] = \int_0^{T_b}\int_0^{T_b} E[n(t)n(\tau)]\psi_k(t)\psi_k(\tau)dtd\tau$$

$$= \frac{N_0}{2}\int_0^{T_b}\int_0^{T_b}\delta(t-\tau)\psi_k(t)\psi_k(\tau)dtd\tau$$

$$= \frac{N_0}{2}\int_0^{T_b}\psi_k^{\ 2}(t)dt = \frac{N_0}{2}$$

- For the first matched filter,

$$\left(\frac{S}{N}\right)_o = \frac{(\sqrt{\mathcal{E}_b})^2}{N_0/2} = \frac{2\mathcal{E}_b}{N_0}$$

- The output of the two matched filters corresponding to the transmitted signal $s_2(t)$ are $(y_1, y_2)=(n_1, \sqrt{\mathcal{E}_b}+n_2)$

Frequency Domain Interpretation of the Matched Filter (8/8)

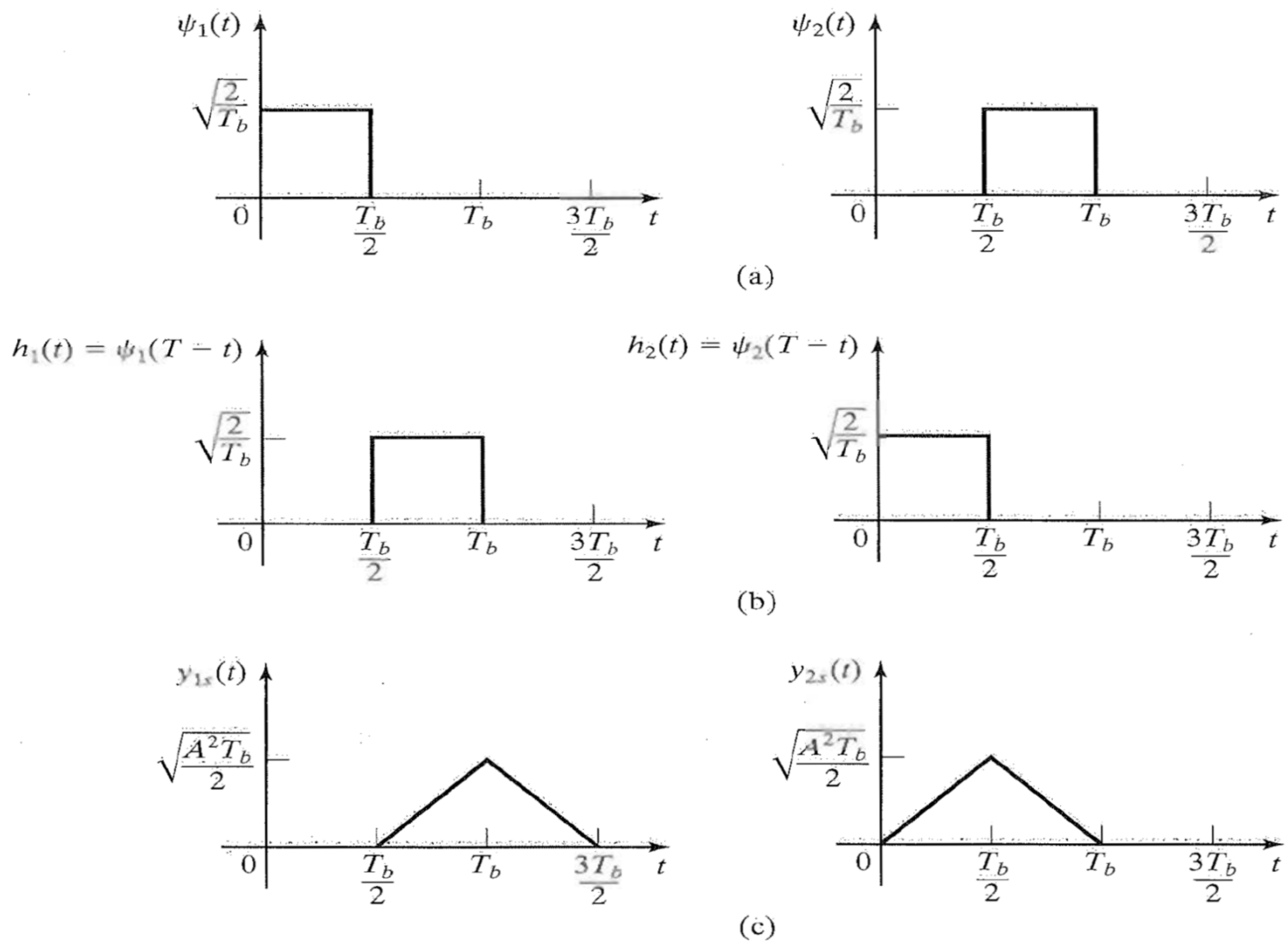


Figure 8.25 Basis functions and matched filter responses for Example 8.3.5.

The Performance of the Optimum Detector for Binary Signals (1/9)

- We describe optimum decision rule employed by the detector to make decisions based on the output from the demodulator
- We assume that the signals received in successive signal intervals are statistically independent so the detector only needs to consider its input in a given bit interval
- First we consider binary antipodal signals. The output of the demodulator in any bit interval is

$$y = s_m + n, \quad m = 1, 2,$$

where $s_m = \pm\sqrt{\mathcal{E}_b}$ and n is a zero-mean Gaussian random variable with variance $N_0/2$

The Performance of the Optimum Detector for Binary Signals (2/9)

- The input to the detector is a scalar. The detector compares y with a threshold α. Determine whether $y>\alpha$ and declares that the signal $s_1(t)$ is transmitted; otherwise, it declares that $s_2(t)$ was transmitted
- For the binary antipodal signals, the average probability of error as a function of the threshold α is

$$P_2(\alpha) = P(s_1)\int_{-\infty}^{\alpha} f(y\,|\,s_1)dy + P(s_2)\int_{\alpha}^{\infty} f(y\,|\,s_2)dy \quad ,$$

where $P(s_1)$ and $P(s_2)$ are the *a priori* probabilities of the two possible transmitted signals

The Performance of the Optimum Detector for Binary Signals (3/9)

- The optimum α to minimize $P_2(\alpha)$ is determined by differentiating $P_2(\alpha)$ with respect to α and setting the derivative to zero. We obtain

$$P(s_1)f(\alpha \mid s_1) - P(s_2)f(\alpha \mid s_2) = 0$$

or equivalently,

$$\frac{f(\alpha \mid s_1)}{f(\alpha \mid s_2)} = \frac{P(s_2)}{P(s_1)} \qquad (8.3.49)$$

- Substitute $s_1 = \sqrt{\mathcal{E}_b}$ and $s_2 = -\sqrt{\mathcal{E}_b}$ and the conditional PDFs into (8.3.49), we have

$$e^{4\alpha\sqrt{\mathcal{E}_b}/N_0} = \frac{P(s_2)}{P(s_1)}$$

The optimum value of the threshold is

$$\alpha^* = \frac{N_0}{4\sqrt{\mathcal{E}_b}} \ln \frac{P(s_2)}{P(s_1)}$$

The Performance of the Optimum Detector for Binary Signals (4/9)

- If $P(s_1) > P(s_2)$, then $\alpha^* < 0$, and if $P(s_2) > P(s_1)$, then $\alpha^* > 0$.
  In practice, the two possible signals are usually equally probable, *i.e.*, the *a priori* probabilities $P(s_1) = P(s_2) = 1/2$, the threshold $\alpha^* = 0$
- For the case of equal *a priori* probabilities, the average probability of error is

$$P_2 = \tfrac{1}{2}\int_{-\infty}^{0} f(y\,|\,s_1)dy + \tfrac{1}{2}\int_{0}^{\infty} f(y\,|\,s_2)dy$$

$$= Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right),$$

where $Q(x)$ is the area under the tail of the normal (Gaussian) probability density function, defined as

$$Q(x) = \tfrac{1}{\sqrt{2\pi}}\int_{x}^{\infty} e^{-u^2/2}du$$

The Performance of the Optimum Detector for Binary Signals (5/9)

- We observe that the probability of error depends only on the signal-to-noise ratio $2\mathcal{E}_b/N_0$ and not on the detailed characteristics of the pulse waveforms
- The average error probability tends to zero exponentially as SNR increases

The Performance of the Optimum Detector for Binary Signals (6/9)

- We now consider binary orthogonal signals. The output of the demodulator is the two-dimensional vector $y=(y_1,y_2)$, where y_1 and y_2 are the output of the two cross-correlators or the two matched filters
- Recall that if the signal $s_1(t)$ is transmitted, the demodulator output are

$$y_1=\sqrt{\mathcal{E}_b}+n_1$$

and

$$y_2=n_2$$

- n_1 and n_2 are statistically independent, zero-mean Gaussian variables with variance $N_0/2$

The Performance of the Optimum Detector for Binary Signals (7/9)

- For the case of equal *a priori* probabilities, *i.e.*, $P(s_1)=P(s_2)=1/2$, the detector that minimizes the average probability of error simply compares y_1 with y_2. If $y_1>y_2$, the detector declares that $s_1(t)$ was transmitted. Otherwise, it declares that $s_2(t)$ was transmitted
- Assume $s_1(t)$ was transmitted, the probability of error is simply the probability that $y_1-y_2<0$.
- y_1 and y_2 are Gaussian with equal variance $N_0/2$ and statistically independent, the difference

$$\begin{aligned} z &= y_1 - y_2 \\ &= \sqrt{\mathcal{E}_b} + n_1 - n_2 \end{aligned}$$

is a Gaussian random variable with mean $\sqrt{\mathcal{E}_b}$ and variance N_0

The Performance of the Optimum Detector for Binary Signals (8/9)

- The probability density function of z is

$$f(z) = \frac{1}{\sqrt{2\pi N_0}} e^{-(z-\sqrt{\mathcal{E}_b})^2 / 2N_0}$$

  and the average probability of error is

$$P_2 = P(z < 0) = \int_{-\infty}^{0} f(z) dz$$
$$= Q\left(\sqrt{\frac{\mathcal{E}_b}{N_0}}\right)$$

- When we compare the average probability of error of binary antipodal signals to that of binary orthogonal signals, we observe that, for the same error probability P_2, the binary antipodal signals require a factor of two (3 dB) less signal energy than orthogonal signals

The Performance of the Optimum Detector for Binary Signals (9/9)

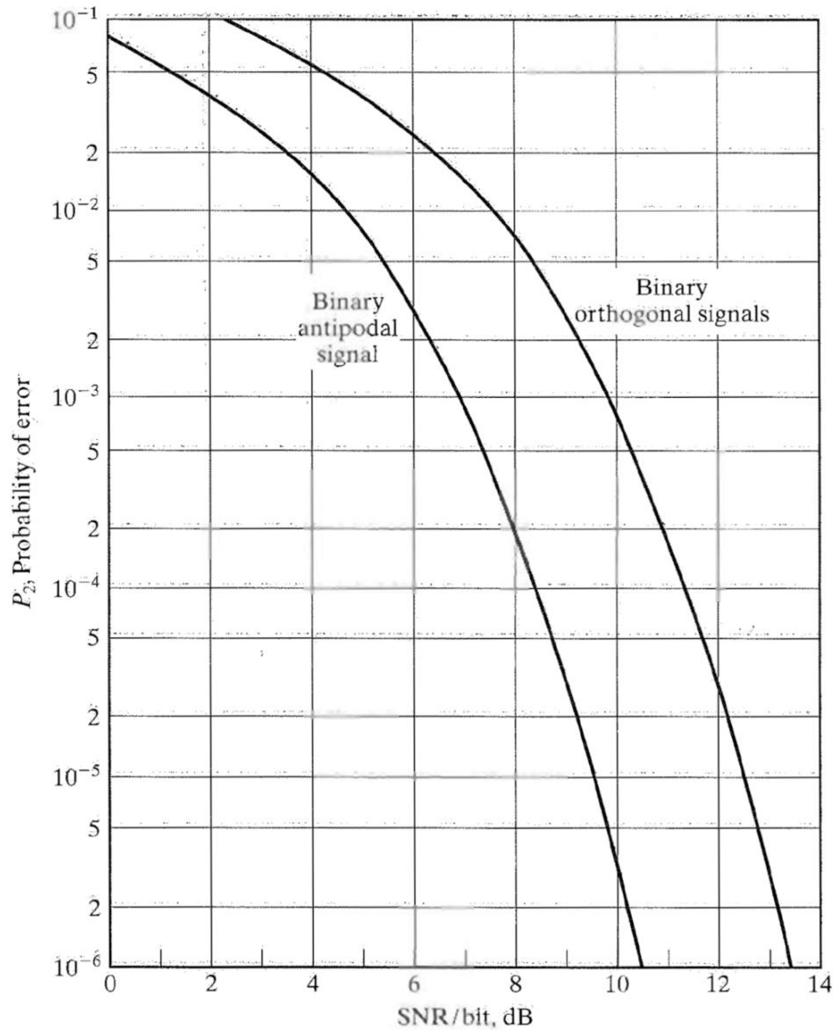


Figure 8.26 Probability of error for binary signals.