Chapter 8 Digital Modulation in an Additive White Gaussian Noise Baseband Channel (I)

Introduction (1/2)

- In this chapter, we consider the transmission of the digital information sequence over communication channels that are characterized as *additive white Gaussian noise* (AWGN) channels
- The AWGN channel is one of the simplest mathematical models for various physical communication channels. Such channels are basically analog channels, which means that the digital information sequence to be transmitted must be mapped into analog signal waveforms

Introduction (2/2)

- In this chapter, we consider baseband channels, i.e., channels having frequency passbands that usually includes zero frequency (f=0)
- When the digital information is transmitted through a baseband channel, there is no need to use a carrier frequency for the transmission of the digitally modulated signals
- There are many communication channels (including telephone channels, radio channels, and satellite channels) that have frequency passbands that are far removed from f=0. These types of channels are called *bandpass channels*

Geometric Representation of Signal Waveforms (1/2)

- In a digital communication system, the modulator input is typically a sequence of binary information digits. The modulator may map each information bit to be transmitted into one of two possible distinct signal waveforms, say $s_1(t)$ or $s_2(t)$
- A zero is represented by the transmitted signal waveform $s_1(t)$, and a one is represented by the transmitted signal waveform $s_2(t)$. This type of digital modulation is called *binary modulation*

Geometric Representation of Signal Waveforms (2/2)

- The modulator may transmit k bits ($k>1$) at a time by employing $M=2^k$ distinct signal waveforms, say $s_m(t)$, $1 \leqq m \leqq M$. This type of digital modulation is called M-ary (nonbinary) modulation
- We develop a vector representation of such digital signal waveforms
- Suppose we have a set of M signal waveforms $s_m(t)$, $1 \leqq m \leqq M$, which are to be used for transmitting information over a communication channel. From the set of M waveforms, we first construct a set of $N \leqq M$ orthonormal waveforms, where N is the dimension of the signal space

Gram-Schmidt Orthogonalization Procedure (1/11)

- We begin with the first waveform $s_1(t)$, which is assumed to have energy $\mathcal{E}_1$. The first waveform of the orthonormal set is constructed as

$$\psi_1(t) = \frac{s_1(t)}{\sqrt{\mathcal{E}_1}}$$

- $\psi_1(t)$ is simply $s_1(t)$ normalized to unit energy
- The second waveform is constructed from $s_2(t)$ by first computing the projection of $s_2(t)$ onto $\psi_1(t)$, which is

$$c_{21} = \int_{-\infty}^{\infty} s_2(t)\psi_1(t)dt$$

- $c_{21}\psi_1(t)$ is subtracted from $s_2(t)$ to yield

$$d_2(t) = s_2(t) - c_{21}\psi_1(t)$$

Gram-Schmidt Orthogonalization Procedure (2/11)

- $d_2(t)$ is orthogonal to $\psi_1(t)$, but it does not possess unit energy
- If $\mathcal{E}_2$ denotes the energy in $d_2(t)$, then the energy-normalized waveform that is orthogonal to $\psi_1(t)$ is

$$\psi_2(t) = \frac{d_2(t)}{\sqrt{\mathcal{E}_2}};$$

$$\mathcal{E}_2 = \int_{-\infty}^{\infty} d_2^{\,2}(t)\,dt$$

- The orthogonalization of the kth function leads to

$$\psi_k(t) = \frac{d_k(t)}{\sqrt{\mathcal{E}_k}}$$

Gram-Schmidt Orthogonalization Procedure (3/11)

- Note

$$d_k(t) = s_k(t) - \sum_{i=1}^{k-1} c_{ki}\psi_i(t),$$

$$\mathcal{E}_k = \int_{-\infty}^{\infty} d_k^{\,2}(t)dt,$$

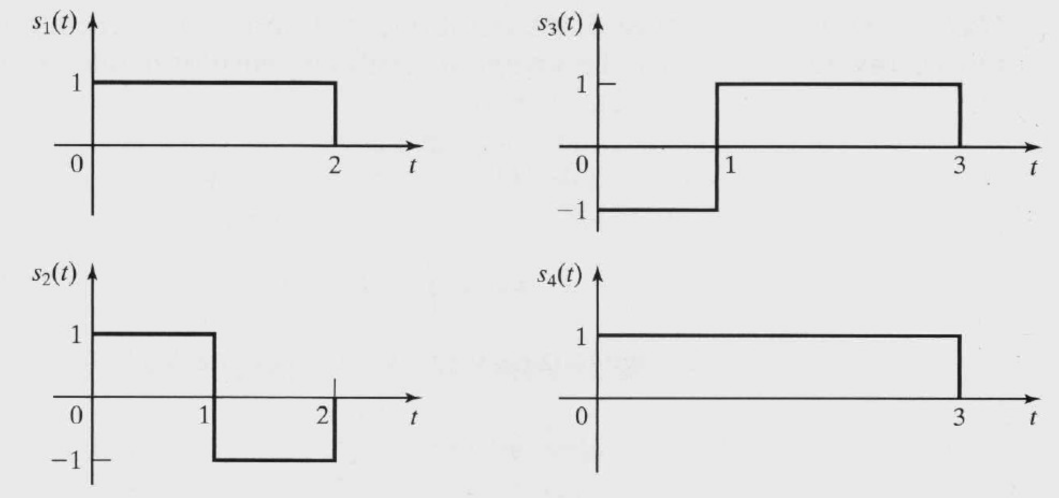
and

$$c_{ki} = \int_{-\infty}^{\infty} s_k(t)\psi_i(t)dt, \qquad i = 1,2,\ldots,k-1$$

- The orthogonalization process is continued until all the M signal waveforms $\{s_m(t)\}$ have been exhausted and $N \leqq M$ orthonormal waveforms have been constructed
- If at any step $d_k(t)=0$, then there will be no new $\psi(t)$; hence, no new dimension is introduced

Gram-Schmidt Orthogonalization Procedure (4/11)

- The dimensionality N of the signal space will be equal to M if all the M signal waveforms are linearly independent, *i.e.*, if none of the signal waveforms is a linear combination of the other signal waveforms
- **Example 8.1.1.** Apply the Gram-Schmidt procedure to the set of four waveforms illustrated in the figure



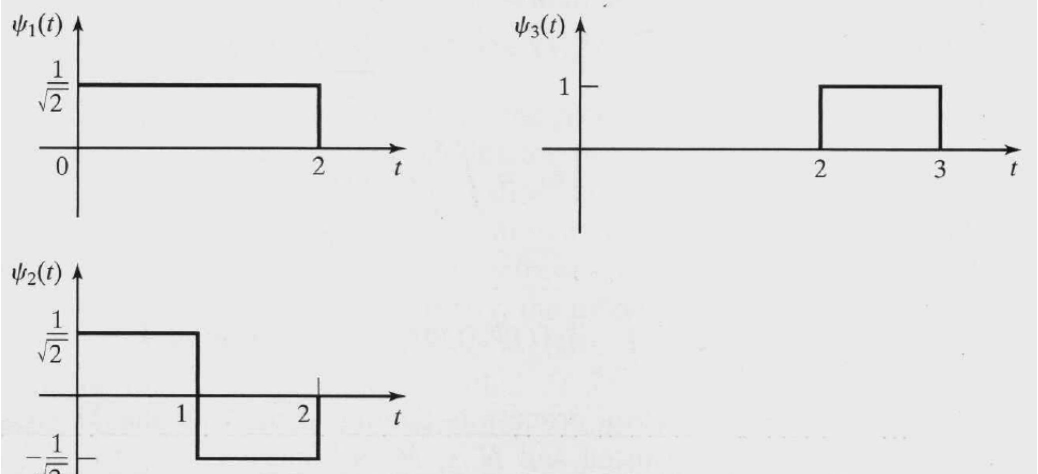
(a) Original signal set

Gram-Schmidt Orthogonalization Procedure (5/11)

- **Example 8.1.1. (Cont'd)** We have $\mathcal{E}_1=2$, so that $\psi_1(t)=s_1(t)/\sqrt{2}$. Next, we observe that $c_{21}=0$, so that $\psi_1(t)$ and $s_2(t)$ are orthogonal. Therefore, $\psi_2(t)=s_2(t)/\sqrt{2}$. To obtain $\psi_3(t)$, we compute $c_{31}=0$ and $c_{32}=-\sqrt{2}$. Hence,

$$d_3(t)=s_3(t)+\sqrt{2}\psi_2(t)$$

- Since $d_3(t)$ has unit energy, it follows that $\psi_3(t)=d_3(t)$



(b) Orthonormal waveforms

Gram-Schmidt Orthogonalization Procedure (6/11)

- **Example 8.1.1. (Cont'd)** Finally, we find that $c_{41}=\sqrt{2}$, $c_{42}=0$, $c_{43}=1$. Hence,

$$d_4(t) = s_4(t) - \sqrt{2}\psi_1(t) - \psi_3(t) = 0$$

- $s_4(t)$ is a linear combination of $\psi_1(t)$ and $\psi_3(t)$; consequently, the dimensionality of the signal set is $N=3$
- Once we have constructed the set of orthonormal waveforms $\{\psi_n(t)\}$, we can express the M signals $\{s_m(t)\}$ as exact linear combinations of the $\{\psi_n(t)\}$

Gram-Schmidt Orthogonalization Procedure (7/11)

- We may write

$$s_m(t) = \sum_{n=1}^{N} s_{mn}\psi_n(t), \qquad m = 1,2,\ldots,M. \qquad (8.1.10)$$

The weighting coefficients are given as

$$s_{mn} = \int_{-\infty}^{\infty} s_m(t)\psi_n(t)dt$$

- Since the basis functions $\{\psi_n(t)\}$ are orthonormal, the energy of each signal waveform is related to the weighting coefficients as follows:

$$\mathcal{E}_m = \int_{-\infty}^{\infty} s_m^{\ 2}(t)dt = \sum_{n=1}^{N} s_{mn}^{\ 2}$$

- On the basis of expression in Eq. (8.1.10), each signal waveform may be represented by the vector

$$\boldsymbol{s}_m = (s_{m1}, s_{m2}, \ldots, s_{mN})$$

Gram-Schmidt Orthogonalization Procedure (8/11)

- The energy of the *m*th signal waveform is simply the square of the length of the vector, or, equivalently, the square of the Euclidean distance from the origin to the point in the N-dimensional space
- We can show that the inner product of two signals is equivalent to the inner product of their vector representations, *i.e.*,

$$\int_{-\infty}^{\infty} s_m(t) s_n(t) dt = \mathbf{s}_m \bullet \mathbf{s}_n$$

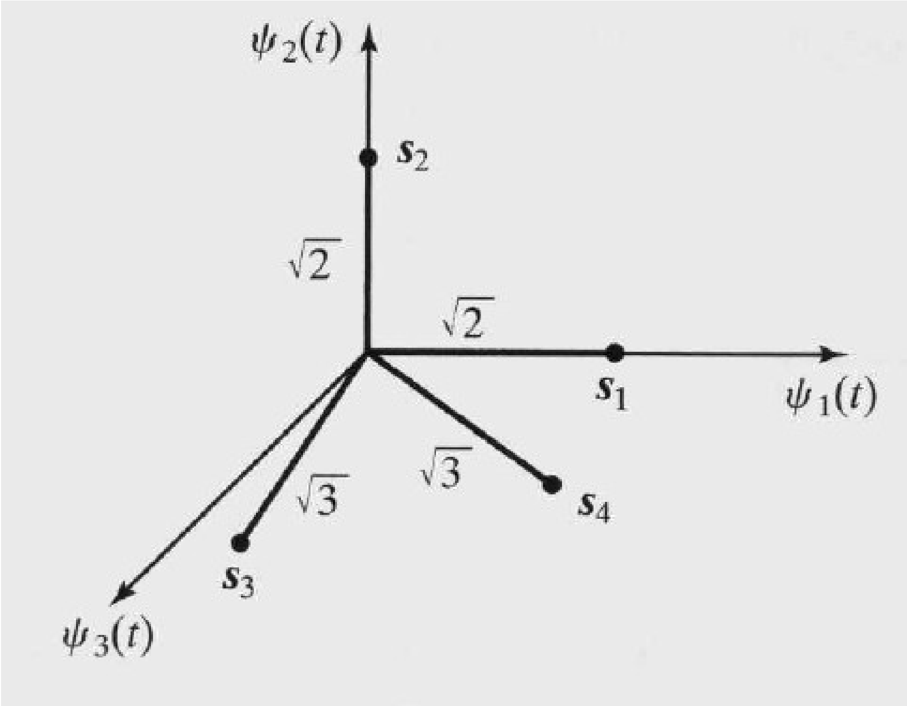
- Any N-dimensional signal can be represented geometrically as a point in the signal space spanned by the N orthonormal functions $\{\psi_n(t)\}$

Gram-Schmidt Orthogonalization Procedure (9/11)

- **Example 8.1.2.** Let us determine the vector representations of the four signals using the orthonormal set of functions derived in Example 8.1.1.
- Since the dimensionality of the signal space is $N=3$, each signal is described by three components, which are obtained by projecting each of the four signal waveforms on the three orthonormal basis functions $\psi_1(t),\ \psi_2(t),\ \psi_3(t)$
- Thus, we obtain $\boldsymbol{s}_1=(\sqrt{2},0,0)$, $\boldsymbol{s}_2=(0,\sqrt{2},0)$, $\boldsymbol{s}_3=(0,-\sqrt{2},1)$ $\boldsymbol{s}_4=(\sqrt{2},0,1)$

Gram-Schmidt Orthogonalization Procedure (10/11)

- **Example 8.1.2. (Cont'd)** These signal vectors are shown in the figure



Gram-Schmidt Orthogonalization Procedure (11/11)

- The set of basis functions $\{\psi_n(t)\}$ obtained by the Gram-Schmidt procedure is not unique. However, the change in the basis functions does not change the dimensionality of the space, N, the lengths (energies) of the signal vectors, or the inner product of any two vectors

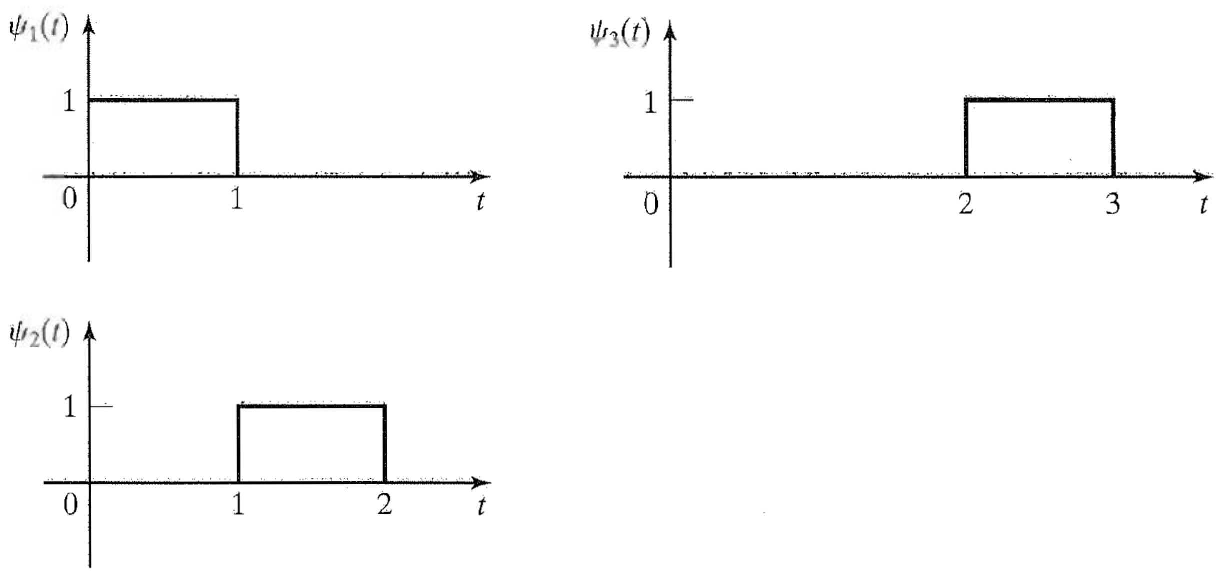


Figure 8.3 Alternate set of basis functions.

Binary Pulse Modulation (1/1)

- We consider two different binary pulse modulation methods: binary pulse amplitude modulation (PAM) and binary pulse position modulation (PPM)
- Assume that the information to be transmitted is a binary sequence that consists of zeros and ones, and occurs at the bit rate R_b bits/sec (bps)

Binary Pulse Amplitude Modulation (1/8)

- Binary PAM is the simplest digital modulation method. The information bit 1 may be represented by a pulse of amplitude A, and the information bit 0 may be represented by a pulse of amplitude $-A$
- Since one signal pulse is the negative of the other, this type of signaling is also called *binary antipodal signaling*

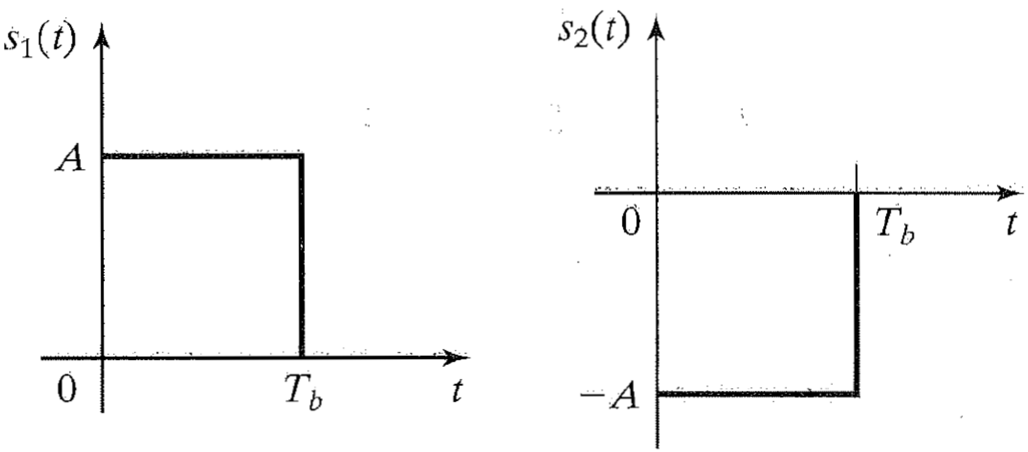


Figure 8.4 Binary PAM signals.

Binary Pulse Amplitude Modulation (2/8)

- Pulses are transmitted at a bit rate $R_b = 1/T_b$ bits/sec. T_b is called the bit interval
- Although the pulses are shown as rectangular, in practical systems the rise time and decay time are nonzero and the pulses are generally smoother. The pulse shape determines the spectral characteristics of the transmitted signal

Binary Pulse Amplitude Modulation (3/8)

- The binary PAM signal waveforms are expressed as

$$s_m(t) = A_m g_T(t), \quad 0 \le t \le T_b, \quad m = 1, 2$$

- A_m takes one of two possible values (A for $m=1$, and $-A$ for $m=2$), and $g_T(t)$ is a rectangular pulse of unit amplitude

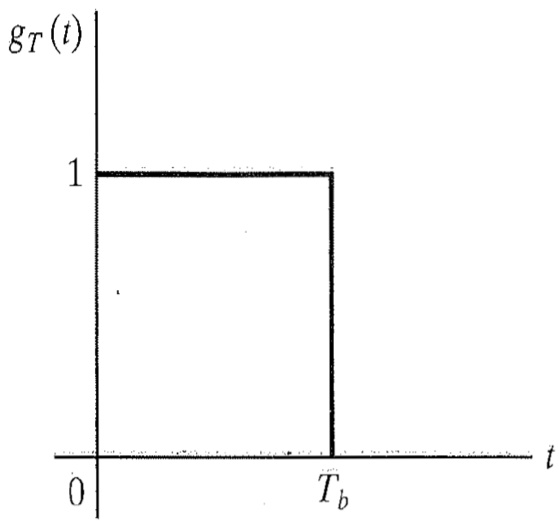


Figure 8.5 A rectangular pulse of unit amplitude and duration T_b.

Binary Pulse Amplitude Modulation (4/8)

- The signal energy in each of the two waveforms is

$$\mathcal{E}_b = \int_0^{T_b} s_m{}^2(t)dt, \quad m = 1,\ 2$$
$$= A^2 \int_0^{T_b} g_T{}^2(t)dt$$
$$= A^2 T_b$$

- Each signal waveform carries one bit of information. We define the signal energy per bit of information as $\mathcal{E}_b$
- We have $A = \sqrt{\mathcal{E}_b / T_b}$
- For binary PAM, the signal waveforms are expressed as

$$s_m(t) = s_m \psi(t), \quad m = 1,\ 2$$

Binary Pulse Amplitude Modulation (5/8)

- $\psi(t)$ is the unit energy rectangular pulse
- $s_1 = \sqrt{\mathcal{E}_b}, \quad s_2 = -\sqrt{\mathcal{E}_b}$
- The binary PAM signal waveforms can be uniquely represented geometrically in one dimension (on the real line) as two vectors and has the amplitude $\sqrt{\mathcal{E}_b}$

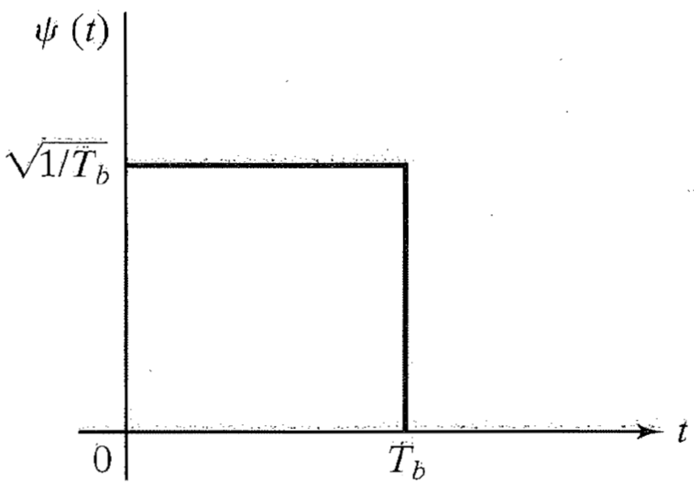


Figure 8.6 Unit energy basis function for binary PAM.

Binary Pulse Amplitude Modulation (6/8)

- We usually omit drawing the vectors from the origin, and we simply display the two endpoints at $\sqrt{\mathcal{E}_b}$ and $-\sqrt{\mathcal{E}_b}$

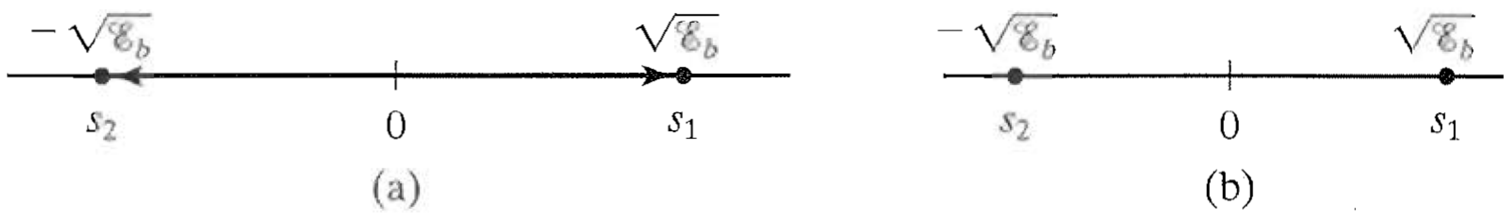


Figure 8.7 Geometric representation of binary PAM.

Binary Pulse Amplitude Modulation (7/8)

- **Example 8.2.1.** Consider two antipodal signal waveforms shown in Fig. 8.8. Show that these signals have exactly the same geometric representations as the two rectangular pulses in Fig. 8.4

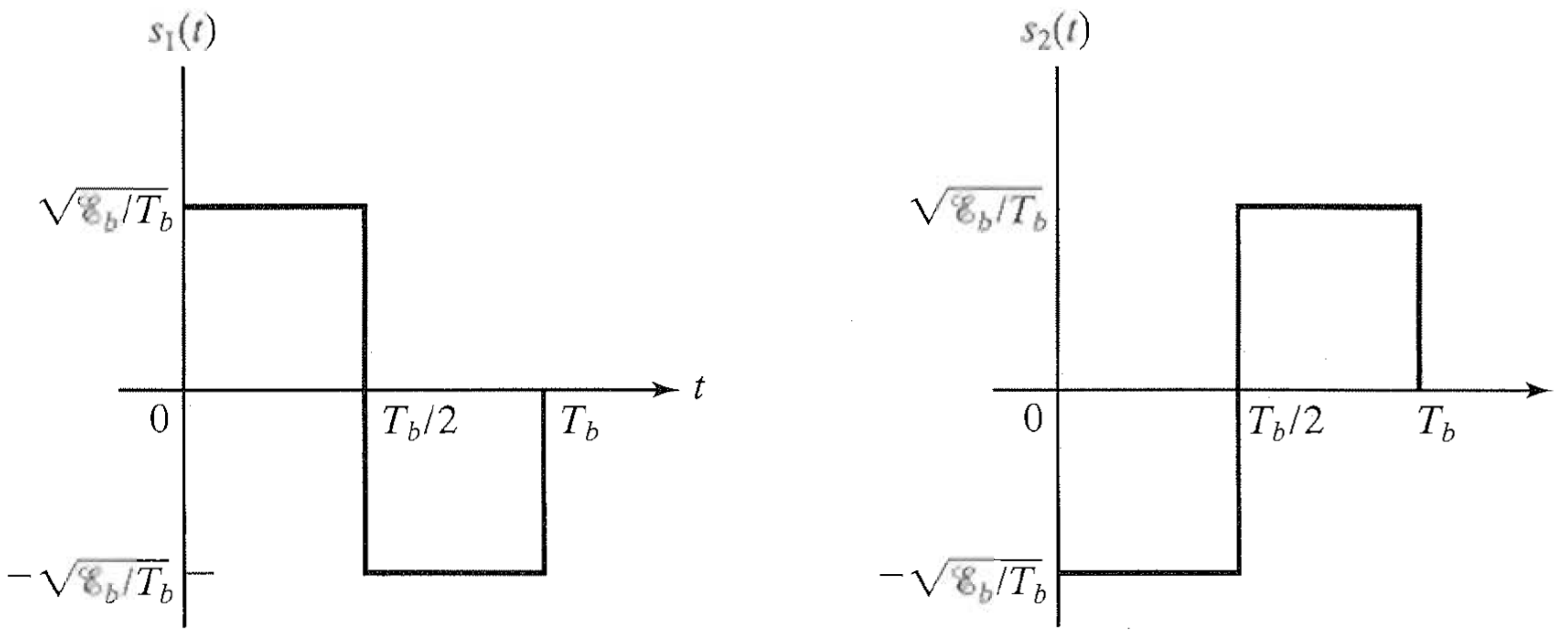


Figure 8.8 Binary antipodal signals in Example 8.2.1.

Binary Pulse Amplitude Modulation (8/8)

- **Example 8.2.1. (Cont'd)** The waveforms of these signals may be represented as

$$s_m(t) = s_m \psi(t), \quad m = 1,\ 2$$

where $\psi(t)$ is the unit energy waveform shown in Fig. 8.9, and $s_1 = \sqrt{\mathcal{E}_b}, \quad s_2 = -\sqrt{\mathcal{E}_b}$. Thus, the two antipodal signal waveforms have exactly the same geometric signal representation as those shown in Fig. 8.4

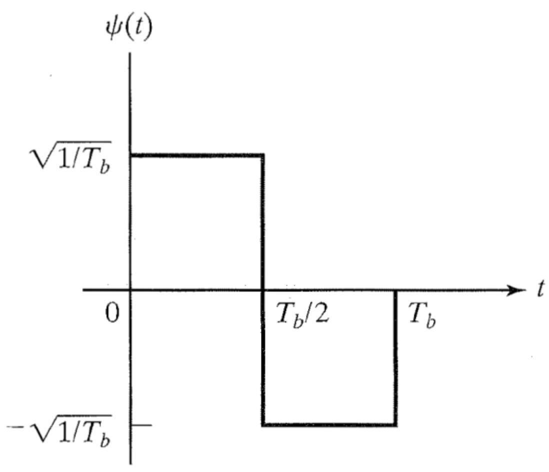


Figure 8.9 Unit energy basis function for the antipodal signals in Figure 8.8.