Chapter 7 Analog-to-Digital Conversion (II)

Scalar Quantization (1/10)

- In scalar quantization, each sample is quantized into one of a finite number of levels, which is then encoded into a binary representation
- The quantization process is a rounding process; each sampled signal point is rounded to the "nearest" value from a finite set of possible quantization levels
- The set of real numbers $\boldsymbol{R}$ is partitioned into N disjoint subset denoted by $\mathcal{R}_k$, $1 \leqq k \leqq N$ (each called a quantization region). Corresponding to each subset $\mathcal{R}_k$, a representation point (or quantization level) $\hat{x}_k$ is chosen, which is usually belongs to $\mathcal{R}_k$

Scalar Quantization (2/10)

- If the sampled signal at time i, x_i belongs to $\mathscr{R}_k$, then it is represented by $\hat{x}_k$, which is the quantized version of x_i
- $\hat{x}_k$ is represented by a binary sequence and transmitted. This step is called encoding
- Since there are N possibilities for the quantized levels, $\log_2 N$ bits are enough to encode these levels into binary sequence. We always assume that N is a power of 2
- The number of bits required to transmit each source output is $R=\log_2 N$ bits
- The price paid for representing (rounding) every sample is the introduction of distortion

Scalar Quantization (3/10)

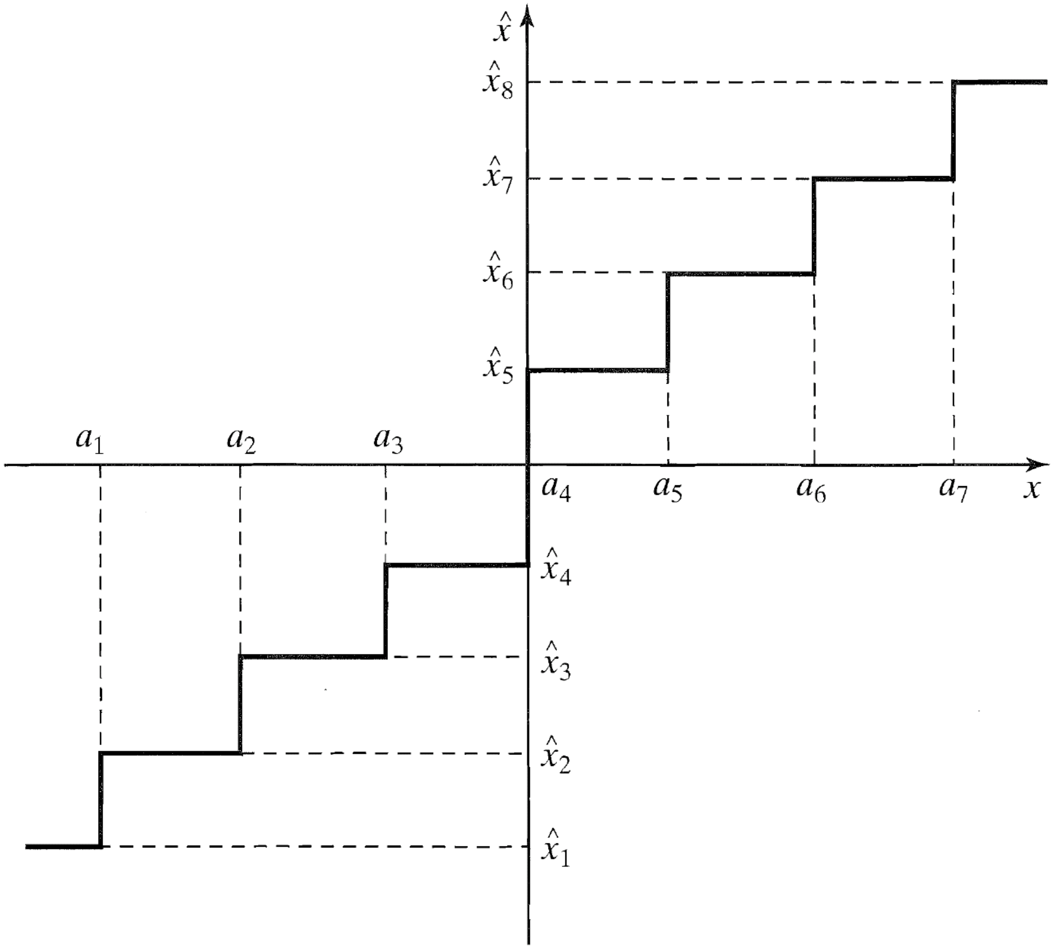

Figure 7.3 Example of an 8-level quantization scheme.

Scalar Quantization (4/10)

- The quantization function Q is defined by

$$Q(x) = \hat{x}_k \quad for\ all\ \ x \in \mathcal{R}_k$$

- We can define the average distortion resulting from quantization. A popular measure of distortion is the squared error distortion defined as $(x - \hat{x}_k)^2$
- x is the sampled signal value and $\hat{x}_k$ is the quantized value, $i.e.,\ \hat{x}_k = Q(x)$
- If we are using the squared error distortion measure, then

$$d(x, \hat{x}) = (x - Q(x))^2 = \tilde{x}^2$$

Let X be a random variable, so are $\hat{X}$ and $\tilde{X}$. The average distortion is given by

$$D = E[d(X, \hat{X})] = E(X - Q(X))^2$$

Scalar Quantization (5/10)

- **Example 7.2.1.** The source $X(t)$ is a stationary Gaussian source with mean zero and power spectral density

$$S_X(f)=\begin{cases}2, & |f|<100\ Hz\\ 0, & otherwise\end{cases}.$$

The source is sampled at the Nyquist rate and each sample is quantized using the 8-level quantizer which is shown in Fig. 7.3. This figure has a_1=-60, a_2=-40, a_3=-20, a_4=-0, a_5=20, a_6=40, a_7=60, and $\hat{x}_1=-70, \hat{x}_2=-50, \hat{x}_3=-30, \hat{x}_4=-10, \hat{x}_5=10, \hat{x}_6=30, \hat{x}_7=50, \hat{x}_8=70.$ What is the resulting distortion and bit rate?

Scalar Quantization (6/10)

- **Example 7.2.1. (Cont'd)** The sampling frequency is f_s=200 Hz. Each sample is a zero-mean Gaussian random variable with variance

$$\sigma^2 = E(X_i^{\,2}) = R_X(\tau)|_{\tau=0} = \int_{-\infty}^{\infty} S_X(f)df = \int_{-100}^{100} 2df = 400$$

- Since each sample is quantized into 8 levels, $\log_2 8=3$ bits are sufficient to represent (encode) the sample; therefore, the resulting rate is

$$R = 3f_s = 600 \ bits/\sec$$

- To find the distortion, we have to evaluate $E(X-\hat{X})^2$ for each sample. We have

$$D = E(X-\hat{X})^2 = \int_{-\infty}^{\infty} (x-Q(x))^2 f_X(x)dx,$$

where $f_X(x)$ is the PDF of the random variable X

Scalar Quantization (7/10)

- **Example 7.2.1. (Cont'd)** We have

$$D=\sum_{i=1}^{8}\int_{R_i}(x-Q(x))^2 f_X(x)dx,$$

or equivalently,

$$D=\int_{-\infty}^{a_1}(x-\hat{x}_1)^2 f_X(x)dx+\sum_{i=2}^{7}\int_{a_{i-1}}^{a_i}(x-\hat{x}_i)^2 f_X(x)dx$$
$$+\int_{a_7}^{\infty}(x-\hat{x}_8)^2 f_X(x)dx,$$

where

$$f_X(x)=\frac{1}{\sqrt{2\pi 400}}e^{-\frac{x^2}{800}}.$$

We obtain $D \doteqdot 33.38$

Scalar Quantization (8/10)

- **Example 7.2.1. (Cont'd)** If we were to use zero bits per source output, then the best strategy would be to set the reconstructed signal to zero. In this case, we have a distortion of $D=E(X\text{-}0)^2=\sigma^2=400$
- This quantization scheme and transmission of 3 bits per source output has enabled us to reduce the distortion to 33.38, which is a factor of 11.98 reduction or 10.78 dB

Scalar Quantization (9/10)

- We choose the mean squared distortion $E(X\text{-}Q(X))^2$, or quantization noise as the measure of performance
- A more meaningful measure of performance is a normalized version of the quantization noise, and it is normalized with respect to the power of the original signal
- The signal-to-quantization noise ratio (SQNR) is defined by

$$SQNR = \frac{E(X^2)}{E(X - Q(X))^2}.$$

$E(X^2)$ is the signal power. $E(X\text{-}Q(X))^2$ is the quantization noise power

Scalar Quantization (10/10)

- **Example 7.2.2.** Determine the SQNR for the quantization scheme given in Example 7.2.1.
- From Example 7.2.1, we have P_X=400 and D=33.38. Therefore,

$$\mathrm{SQNR} = P_X / D = 11.98 \fallingdotseq 10.78 \text{ dB}$$

Uniform Quantization (1/5)

- Uniform quantizers are the simplest examples of scalar quantizers
- The entire real line is partitioned into N regions
- All regions except $\Re_1$ and $\Re_N$ are of equal length Δ. This means that for all $1 \leqq i \leqq N$-2, we have $a_{i+1}-a_i=\Delta$
- It is further assumed that quantization levels are at a distance of $\Delta/2$ from the boundaries $a_1, a_2, \ldots, a_{N-1}$

Uniform Quantization (2/5)

- In a uniform quantizer, the mean squared error distortion is given by

$$D = \int_{-\infty}^{a_1} (x - (a_1 - \Delta/2))^2 f_X(x)dx + \sum_{i=1}^{N-2} \int_{a_1+(i-1)\Delta}^{a_1+i\Delta} (x - (a_1 + i\Delta - \Delta/2))^2 f_X(x)dx + \int_{a_1+(N-2)\Delta}^{\infty} (x - (a_1 + (N-2)\Delta + \Delta/2))^2 f_X(x)dx. \quad (7.2.7)$$

- *D* is a function of two design parameters, a_1 and Δ. In order to design the optimal uniform quantizer, we have to differentiate *D* with respect to these variables and find the values that minimize *D*

Uniform Quantization (3/5)

- Minimization of distortion is generally a tedious task and is done mainly by numerical techniques. Table 7.1 gives the optimal quantization level spacing for a zero-mean unit-variance Gaussian random variable. The last column in the table gives the entropy of the quantizer
- Entropy is defined as

$$H(\hat{x}) = \sum_i p_i \log_2\left(\frac{1}{p_i}\right)$$

, p_i denotes the probability of the *i*th region

Uniform Quantization (4/5)

TABLE 7.1 OPTIMAL UNIFORM QUANTIZER FOR A GAUSSIAN SOURCE

Number Output Levels N	Output-level Spacing Δ	Mean-squared Error D	Entropy $H(\hat{x})$
1	—	1.000	0.0
2	1.596	0.3634	1.000
3	1.224	0.1902	1.536
4	0.9957	0.1188	1.904
5	0.8430	0.08218	2.183
6	0.7334	0.06065	2.409
7	0.6508	0.04686	2.598
8	0.5860	0.03744	2.761
9	0.5338	0.03069	2.904
10	0.4908	0.02568	3.032
11	0.4546	0.02185	3.148
12	0.4238	0.01885	3.253
13	0.3972	0.01645	3.350
14	0.3739	0.01450	3.440
15	0.3534	0.01289	3.524
16	0.3352	0.01154	3.602
17	0.3189	0.01040	3.676
18	0.3042	0.009430	3.746
19	0.2909	0.008594	3.811
20	0.2788	0.007869	3.874

Uniform Quantization (5/5)

21	0.2678	0.007235	3.933
22	0.2576	0.006678	3.990
23	0.2482	0.006185	4.045
24	0.2396	0.005747	4.097
25	0.2315	0.005355	4.146
26	0.2240	0.005004	4.194
27	0.2171	0.004687	4.241
28	0.2105	0.004401	4.285
29	0.2044	0.004141	4.328
30	0.1987	0.003905	4.370
31	0.1932	0.003688	4.410
32	0.1881	0.003490	4.449
33	0.1833	0.003308	4.487
34	0.1787	0.003141	4.524
35	0.1744	0.002986	4.560
36	0.1703	0.002843	4.594