Chapter 7 Analog-to-Digital Conversion (I)

Introduction (1/2)

- To convert an analog signal to a digital signal, three operations must be completed
  - Sampling: the analog signal has to be sampled, so that we can obtain a *discrete-time continuous-valued* signal
  - Quantization: the sampled values which can take an infinite number of values are quantized, *i.e.*, rounded to a finite number of values. We have a *discrete-time, discrete-amplitude* signal
  - Encoding: a sequence of bits (ones and zeros) are assigned to different output of the quantizer

Introduction (2/2)

- The possible outputs of the quantizer are finite, each sample of the signal can be represented by a finite number of bits. If the quantizer has $256=2^8$ possible levels, they can be represented by 8 bits

Sampling of Signals and Signal Reconstruction from Samples (1/4)

- The sampling theorem is one of the most important results in the analysis of signals; it has widespread applications in communications and signal processing
- Many modern signal-processing techniques and the whole family of digital communications methods are based on the validity of this theorem and the insight provided by it
- This theorem together with results from signal quantization techniques, provides a bridge that connects the analog world to digital techniques

Sampling of Signals and Signal Reconstruction from Samples (2/4)

- Consider two signals $x_1(t)$ and $x_2(t)$ shown in Fig. 7.1

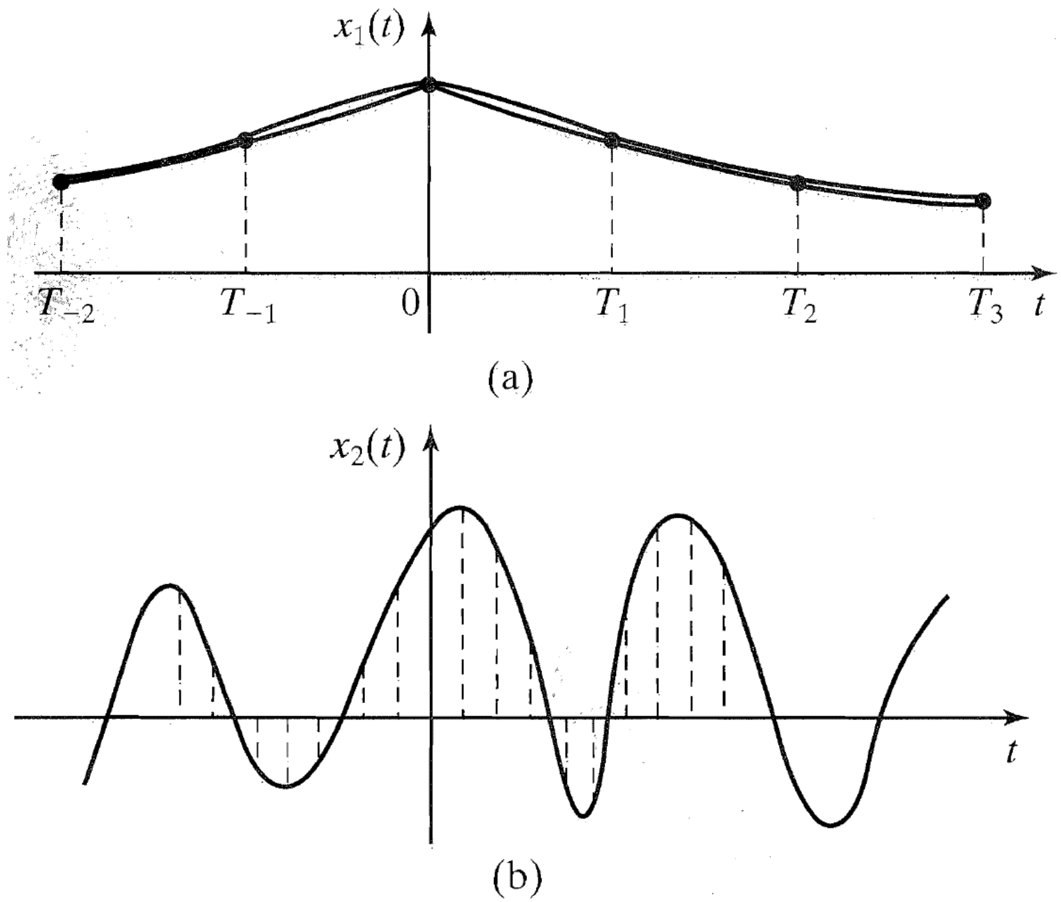

Figure 7.1 Sampling of signals.

Sampling of Signals and Signal Reconstruction from Samples (3/4)

- The first signal $x_1(t)$ is a smooth signal, it varies very slowly; therefore, its main frequency content is at low frequencies
- x_2(t) is a signal with rapid changes due to the presence of high-frequency components
- It is obvious that the sampling interval for the signal $x_1(t)$ can be much larger than the sampling interval necessary to reconstruct signal $x_2(t)$ with comparable distortion
- The sampling interval for the signals of smaller bandwidth can be made larger, or the sampling frequency can be made smaller

Sampling of Signals and Signal Reconstruction from Samples (4/4)

- The sampling theorem states two facts:

1. If the signal $x(t)$ is bandlimited to W, *i.e.*, if $X(f)=0$ for $|f| \geqq W$, then it is sufficient to sample at intervals $T_s = 1/2W$
2. If we are allowed to employ more sophisticated interpolating signals than linear interpolation, we are able to recover the exact original signal from the samples, as long as condition 1 is satisfied

- The sampling theorem provides both a method to reconstruct the original signal from the sampled values, and it also gives a precise upper bound on the sampling interval required for distortionless reconstruction

Sampling Theorem (1/13)

- Let the signal $x(t)$ have a bandwidth W, *i.e.*, let $X(f)=0$ for $|f| \geqq W$. Let $x(t)$ be sampled at multiples of some basic sampling interval T_s, where $T_s \leqq 1/2W$, to yield the sequence $\{x(nT_s)\}_{n=-\infty}^{+\infty}$. Then it is possible to reconstruct the original signal $x(t)$ from the sampled values by the reconstruction formula

$$x(t) = \sum_{n=-\infty}^{\infty} 2W' T_s x(nT_s) \operatorname{sinc}[2W'(t-nT_s)]$$

where W' is any arbitrary number that satisfies the condition

$$W \leq W' \leq \tfrac{1}{T_s} - W$$

Sampling Theorem (2/13)

- In the special case where $T_s = 1/2W$, we will have $W' = W = 1/2T_s$ and the reconstruction relation simplifies to

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\mathrm{sinc}\left(\frac{t}{T_s} - n\right)$$

- The sinc function is defined as

$$\mathrm{sinc}(t) \overset{\Delta}{=} \begin{cases} \dfrac{\sin(\pi t)}{\pi t}, & t \neq 0 \\ 1, & t = 0 \end{cases}$$

Sampling Theorem (3/13)

- **Proof**. Let $x_\delta(t)$ denote the result of sampling the original signal by impulses at nT_s time instant. Then

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s).$$

We can write

$$x_\delta(t) = x(t)\sum_{n=-\infty}^{\infty} \delta(t - nT_s),$$

where we have used the property that $x(t)\delta(t\text{-}nT_s)=x(nT_s)\delta(t\text{-}nT_s)$

- If we take the Fourier transform of both sides of the preceding relation, we obtain

$$X_\delta(f) = X(f) \bigstar \mathscr{F}\left[\sum_{n=-\infty}^{\infty} \delta(t - nT_s)\right] \qquad (7.1.4)$$

Sampling Theorem (4/13)

- Using Table 2.1 to find the Fourier transform of $\sum_{n=-\infty}^{\infty}\delta(t-nT_s)$, we obtain

$$\mathcal{F}\left[\sum_{n=-\infty}^{\infty}\delta(t-nT_s)\right]=\frac{1}{T_s}\sum_{n=-\infty}^{\infty}\delta(f-\frac{n}{T_s}). \qquad (7.1.5)$$

Substituting Eq. (7.1.5) into Eq. (7.1.4), we obtain

$$X_\delta(f)=X(f)\bigstar\frac{1}{T_s}\left[\sum_{n=-\infty}^{\infty}\delta(f-\frac{n}{T_s})\right]$$

$$=\frac{1}{T_s}\left[\sum_{n=-\infty}^{\infty}X(f-\frac{n}{T_s})\right]$$

- In the last step, we have employed the convolution property of the impulse signal, which states $X(f)\bigstar\delta(f\text{-}n/T_s)=X(f\text{-}n/T_s)$

Sampling Theorem (5/13)

- Fig. 7.2 shows a plot of $X_\delta(f)$

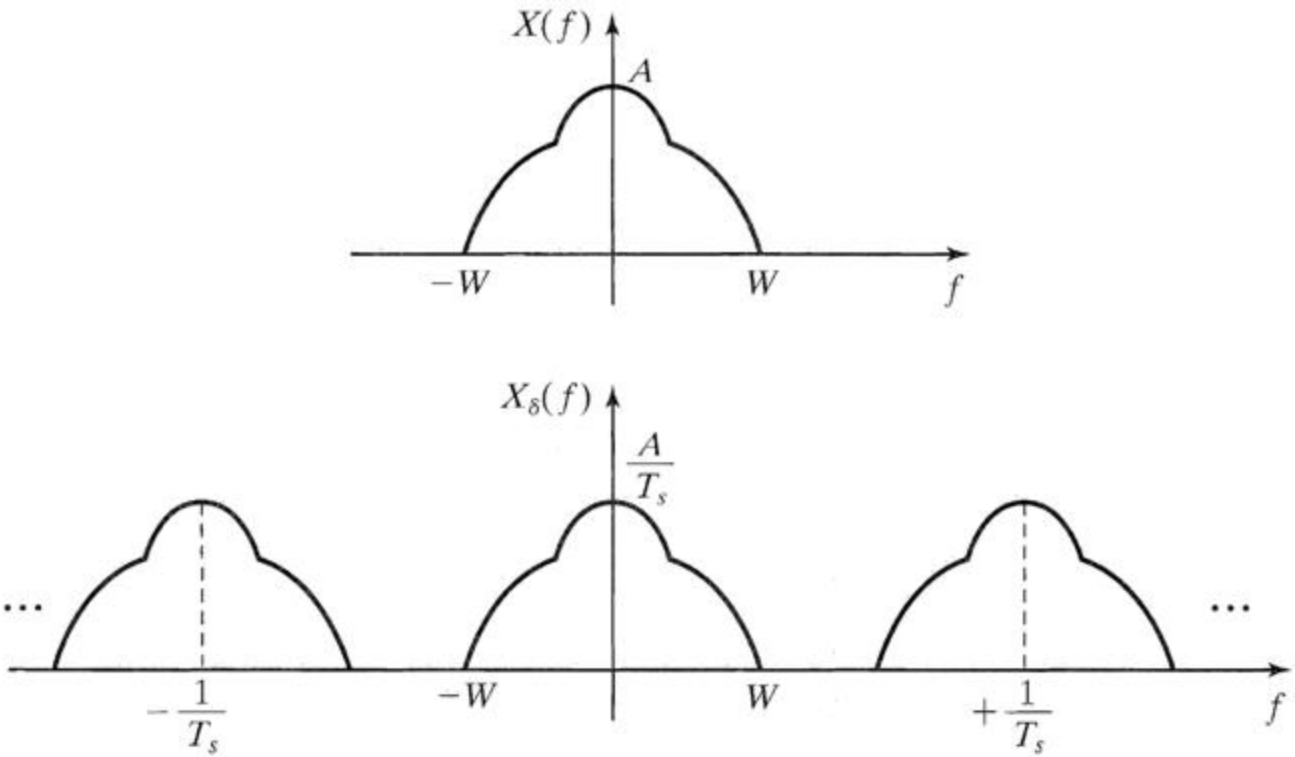

Figure 7.2 Frequency-domain representation of the sampled signal.

Sampling Theorem (6/13)

- If $T_s > 1/2W$, then the replicated spectrum of $x(t)$ overlaps and reconstruction of the original signal is not possible. This type of distortion is known as *aliasing error* or *aliasing distortion*
- If $T_s \leq 1/2W$, no overlap occurs. It is sufficient to filter the sampled signal through a lowpass filter with the frequency response characteristic
  - $H(f) = T_s$ for $|f| < W$
  - $H(f) = 0$ for $|f| \geq 1/T_s - W$
- One obvious choice for $H(f)$ is

$$H(f) = T_s \Pi\left(\frac{f}{2W'}\right),$$

where $W \leqq W' < 1/T_s - W$

Sampling Theorem (7/13)

- With this choice, we have

$$X(f) = X_{\delta}(f) T_s \Pi\left(\frac{f}{2W'}\right),$$

- Taking the inverse Fourier transform of both sides, we obtain

$$\begin{aligned} x(t) &= x_{\delta}(t) \star 2W' T_s \operatorname{sinc}\left(2W' t\right) \\ &= \left(\sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)\right) \star 2W' T_s \operatorname{sinc}\left(2W' t\right) \\ &= \sum_{n=-\infty}^{\infty} 2W' T_s x(nT_s) \operatorname{sinc}(2W'(t - nT_s)) \end{aligned}$$

- This relation shows that if we use sinc functions for interpolation of the sampled values, we can perfectly reconstruct the original signal

Sampling Theorem (8/13)

- The sampling rate $f_s = 2W$ is the minimum at which no aliasing occurs. This sampling rate is known as the *Nyquist sampling rate*. The sampling period is $T_s = 1/2W$.
- If sampling is done at the Nyquist rate, then the only choice for the reconstruction filter is an ideal lowpass filter. In this case,

$$
\begin{aligned}
x(t) &= \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) \operatorname{sinc}(2Wt - n) \\
&= \sum_{n=-\infty}^{\infty} x(nT_s) \operatorname{sinc}\left(\frac{t}{T_s} - n\right)
\end{aligned}
$$

Sampling Theorem (9/13)

- In practical systems, sampling is done at a rate higher than the Nyquist rate. This allows for the reconstruction filter to be realizable and easier to build
- In such cases, the distance between two adjacent replicated spectra in the frequency domain, *i.e.*, $(1/T_s - W) - W = f_s - 2W$, is known as the *guard band*
- In systems with a guard band, we have $f_s = 2W + W_G$, where W is the bandwidth of the signal, W_G is the guard band, and f_s is the sampling frequency

Sampling Theorem (10/13)

- Note that there exists a strong similarity between our development of the sampling theorem and our previous development of the Fourier transform for periodic signals (or Fourier series)
- This similarity is a consequence of the duality between the time and frequency domains and the fact that both the Fourier-series expansion and the reconstruction from samples are orthogonal expansions, one in terms of the exponential signals and the other in terms of the sinc functions

Sampling Theorem (11/13)

- **Example 7.1.1.** We have assumed that samples are taken at multiples of T_s. What happens if we sample regularly with T_s as the sampling interval, but the first sample is taken at some $0<t_0<T_s$?
- We define a new signal $y(t)=x(t+t_0)$. Then $y(t)$ is bandlimited with $Y(f)=e^{j2\pi f t_0}X(f)$ and the samples of $y(t)$ at $\{kT_s\}_{k=-\infty}^{\infty}$ are equal to the samples of $x(t)$ at $\{t_0+kT_s\}_{k=-\infty}^{\infty}$
- Applying the sampling theorem to the reconstruction of $y(t)$, we have

$$y(t) = \sum_{k=-\infty}^{\infty} y(kT_s)\operatorname{sinc}(2W(t-kT_s))$$
$$= \sum_{k=-\infty}^{\infty} x(t_0+kT_s)\operatorname{sinc}(2W(t-kT_s))$$

Sampling Theorem (12/13)

- **Example 7.1.1. (Cont'd)**
- Hence,

$$x(t+t_0)=\sum_{k=-\infty}^{\infty}x(t_0+kT_s)\operatorname{sin}c(2W(t-kT_s))$$

- Substituting $t=t-t_0$, we obtain the following important interpolation relation:

$$x(t)=\sum_{k=-\infty}^{\infty}x(t_0+kT_s)\operatorname{sin}c(2W(t-t_0-kT_s))$$

$$=\sum_{k=-\infty}^{\infty}x(t_0+kT_s)\operatorname{sin}c(2W(t-(t_0+kT_s)))$$

Sampling Theorem (13/13)

- **Example 7.1.2.** A bandlimited signal has a bandwidth equal to 3400 Hz. What sampling rate should be used to guarantee a guard band of 1200 Hz?
- We have

$$f_s = 2W + W_G;$$

therefore,

$$f_s = 2 \times 3400 + 1200 = 8000$$

- After sampling the continuous-time signal, it is transformed to a discrete-time signal. After this step, we have samples taken at discrete times, but the amplitude of these samples is still continuous