Chapter 5 Probability and Random Processes (III)

Gaussian and White Processes (1/1)

- Thermal noise in electronic devices, which is produced by the random movement of electrons due to thermal agitation, can be closely modeled by a Gaussian process
- Gaussian processes provide rather good models for some information sources
- Some interesting properties of the Gaussian processes, which will be discussed in this section, make these processes mathematically tractable and easy to use

Preliminary Knowledge to Understand Jointly Gaussian Process (1/2)

- Jointly Gaussian or binormal random variables X_1 and X_2 are distributed according to a joint PDF of the form

$$f_{X_1,X_2}(x_1,x_2)=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\{-\frac{1}{2(1-\rho^2)}[\frac{(x_1-m_1)^2}{\sigma_1^2}+\frac{(x_2-m_2)^2}{\sigma_2^2}-\frac{2\rho(x_1-m_1)(x_2-m_2)}{\sigma_1\sigma_2}]\}$$

where m_1, m_2, $\sigma_1{}^2$, $\sigma_2{}^2$ are the mean and variance of X_1 and X_2, respectively. ρ is their correlation coefficient

- The definition of two jointly Gaussian random variables can be extended to more random variables. For instance, X_1, X_2, and X_3 are jointly Gaussian if the joint PDF follows the jointly Gaussian PDF

Preliminary Knowledge to Understand Jointly Gaussian Process (2/2)

- The multivariate Gaussian random variables have density

$$f_X(x_1,\cdots,x_k)=\frac{1}{\sqrt{(2\pi)^k|\Sigma|}}\exp\{-\tfrac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\},$$

where $\boldsymbol{x}$ is a k-dimensional column vector, $\boldsymbol{\Sigma}$ is the symmetric covariance matrix, $|\boldsymbol{\Sigma}|\equiv\det\boldsymbol{\Sigma}$ is the determinant of $\boldsymbol{\Sigma}$

- The equation above reduces to that of the univariate normal distribution if $\boldsymbol{\Sigma}$ is a 1×1 matrix (*i.e.*, a single real number)

Gaussian Processes (1/5)

- A random process $X(t)$ is a Gaussian process if for all n and all $(t_1, t_2, \ldots, t_n)$, the random variables $\{X(t_i)\}_{i=1}^{n}$ have a jointly Gaussian density function
- At any time instant t_0, the random variable $X(t_0)$ is Gaussian; at any two points t_1, t_2, random variables $(X(t_1), X(t_2))$ are distributed according to a two-dimensional jointly Gaussian density function
- **Example 5.3.1.** Let $X(t)$ be a zero-mean WSS Gaussian random variable with the power spectral density $S_X(f) = 5\,\Pi(f/1000)$. Determine the probability density function of the random variable $X(3)$

Gaussian Processes (2/5)

- **Example 5.3.1. (Cont'd)** Since $X(t)$ is a Gaussian random process, the probability density function of random variable $X(t)$ at any value of t is Gaussian.
- $X(3) \sim \mathcal{N}(m, \sigma^2)$
- Since the process is zero mean, at any time instance t, we have $E[X(t)]=0$; $m=E[X(3)]=0$
- Note that

$$\begin{aligned}\sigma^2 &= VAR[X(3)] \\ &= E[X^2(3)] - (E[X(3)])^2 \\ &= E[X^2(3)] \\ &= R_X(0) \\ &= \int_{-\infty}^{\infty} S_X(f)\,df \\ &= 5000\end{aligned}$$

Gaussian Processes (3/5)

- **Example 5.3.1. (Cont'd)** Therefore, $X(3) \sim \mathcal{N}(0, 5000)$, or the density function of $X(3)$ is

$$f_X(x) = \frac{1}{\sqrt{10000\pi}} e^{-\frac{x^2}{10000}}$$

- The random processes $X(t)$ and $Y(t)$ are *jointly Gaussian* if for all n, m and all $(t_1, t_2, \ldots, t_n)$ and $(\tau_1, \tau_2, \ldots, \tau_m)$, the random vector $(X(t_1), X(t_2), \ldots, X(t_n), Y(\tau_1), Y(\tau_2), \ldots, Y(\tau_m))$ is distributed according to an $n+m$ dimensional jointly Gaussian distribution
- If $X(t)$ and $Y(t)$ are jointly Gaussian, then each of them is individually Gaussian; but the converse is not always true. That is, two individually Gaussian random processes are not always jointly Gaussian

Gaussian Processes (4/5)

- If the Gaussian process $X(t)$ is passed through an LTI system, then the output process $Y(t)$ will also be a Gaussian process. Moreover, $X(t)$ and $Y(t)$ will be jointly Gaussian processes
- For jointly Gaussian processes, uncorrelatedness and independence are equivalent
- **Example 5.3.2.** $Y(t)$ is the output process of a differentiator and $X(t)$ is the system input defined in Example 5.3.1. Determine the probability density function of $Y(3)$.
- Since a differentiator is an LTI system, $Y(t)$ is a Gaussian process
- We can show that $m_Y=0$

Gaussian Processes (5/5)

- **Example 5.3.2. (Cont'd)** We have

$$\begin{aligned}\sigma_Y^2 &= \int_{-\infty}^{\infty} S_Y(f)df \\ &= \int_{-500}^{500} 5|j2\pi f|^2 \, df \\ &= \int_{-500}^{500} 5 \times 4\pi^2 f^2 df \\ &\approx 1.64 \times 10^{10}\end{aligned}$$

- Since $m_Y=0$ and $\sigma_Y^2 = 1.64 \times 10^{10}$, we have $Y(3) \sim \mathcal{N}(0, 1.64 \times 10^{10})$

White Processes (1/5)

- *White process* is used to denote processes in which all frequency components appear with equal power, *i.e.*, the power spectral density is a constant for all frequencies
- This parallels the notion of "white light" in which all colors exist
- A process $X(t)$ is called a white process if it has a flat spectral density, *i.e.*, if $S_X(f)$ is a constant for all f

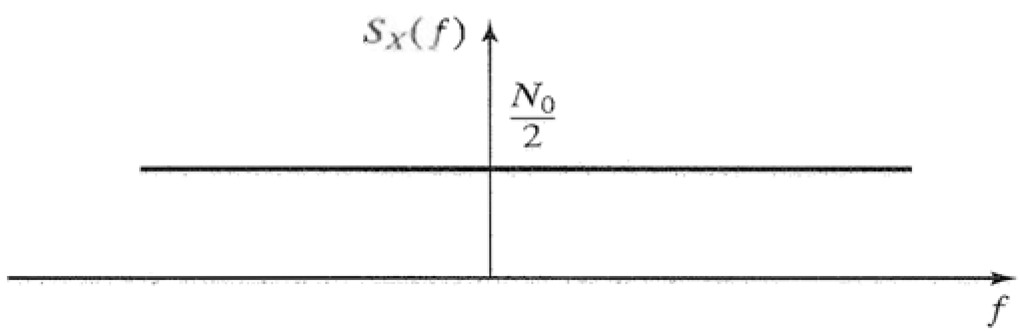

Figure 5.19 Power spectrum of a white process.

White Processes (2/5)

- If we find the power content of a white process using $S_X(f)=C$, a constant, we will have

$$P_X = \int_{-\infty}^{\infty} S_X(f)df = \int_{-\infty}^{\infty} Cdf = \infty$$

- No real physical process can have infinite power; therefore, a white process is not a meaningful physical process
- Quantum mechanical analysis of the thermal noise shows that it has a power spectral density given by

$$S_n(f) = \frac{\hbar f}{2(e^{\frac{\hbar f}{kT}} - 1)}$$

White Processes (3/5)

- $\hbar$ denotes Planck's constant (6.6×10^{-34} J-sec)
- k is Boltzmann's constant (1.38×10^{-23} J/K)
- T is the temperature in degrees Kelvin
- Thermal noise, though not precisely white, for all practical purposes can be modeled as a white process with a power spectrum equal to $kT/2$
- kT is usually denoted by N_0

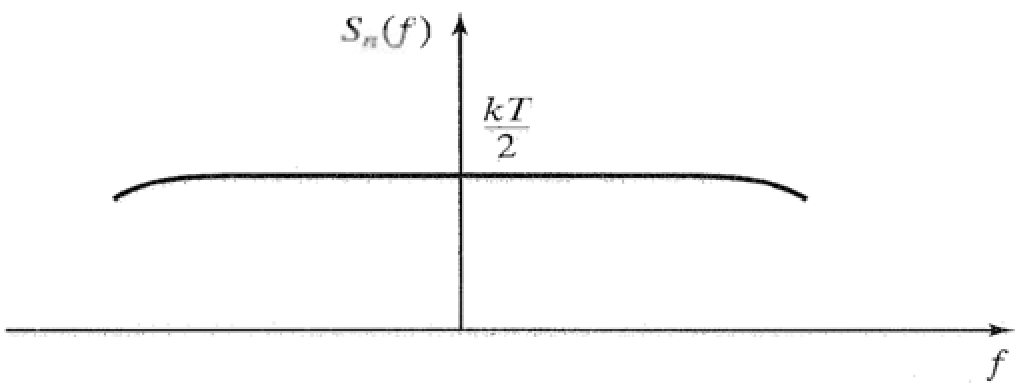

Figure 5.20 Power spectrum of thermal noise.

White Processes (4/5)

- The power spectral density of thermal noise is usually given as $S_n(f)=N_0/2$. It is sometimes referred to as the *two-sided power spectral density*, emphasizing that this spectrum extends to both positive and negative frequencies
- The autocorrelation function for a white process is

$$R_n(\tau) = \mathcal{F}^{-1}\left[\frac{N_0}{2}\right] = \frac{N_0}{2}\delta(\tau)$$

- If we sample a white process at two points t_1 and t_2 ($t_1 \neq t_2$), the resulting random variables will be uncorrelated
- If a random process is white and also Gaussian, any pair of random variables $X(t_1)$, $X(t_2)$, where $t_1 \neq t_2$, will also be independent (uncorrelatedness and independent are equivalent)

White Processes (5/5)

- In subsequent chapters, we assume the following properties:
  - Thermal noise is a WSS process
  - Thermal noise is a zero-mean process
  - Thermal noise is a Gaussian process
  - Thermal noise is a white process with a power spectral density $S_n(f) = kT/2$

Filtered Noise Processes (1/5)

- For a bandpass process the power spectral density is located away from the zero frequency and is mainly located around some frequency f_c, which is far from zero and larger than the bandwidth of the process
- A bandpass process can be expressed in terms of the in-phase and quadrature components
- Consider the process $X(t)$ that is the output of an ideal bandpass filter located at frequencies around f_c. The bandwidth of the filter can be either W or $2W$

Filtered Noise Processes (2/5)

- Consider two filters. The first one has a bandwidth $2W$ and its transfer function of the form

$$H_1(f) = \begin{cases} 1, & |f - f_c| \leq W \\ 0, & otherwise \end{cases}$$

- The other one has a bandwidth W and its transfer function of the form

$$H_2(f) = \begin{cases} 1, & f_c \leq |f| \leq f_c + W \\ 0, & otherwise \end{cases}$$

Filtered Noise Processes (3/5)

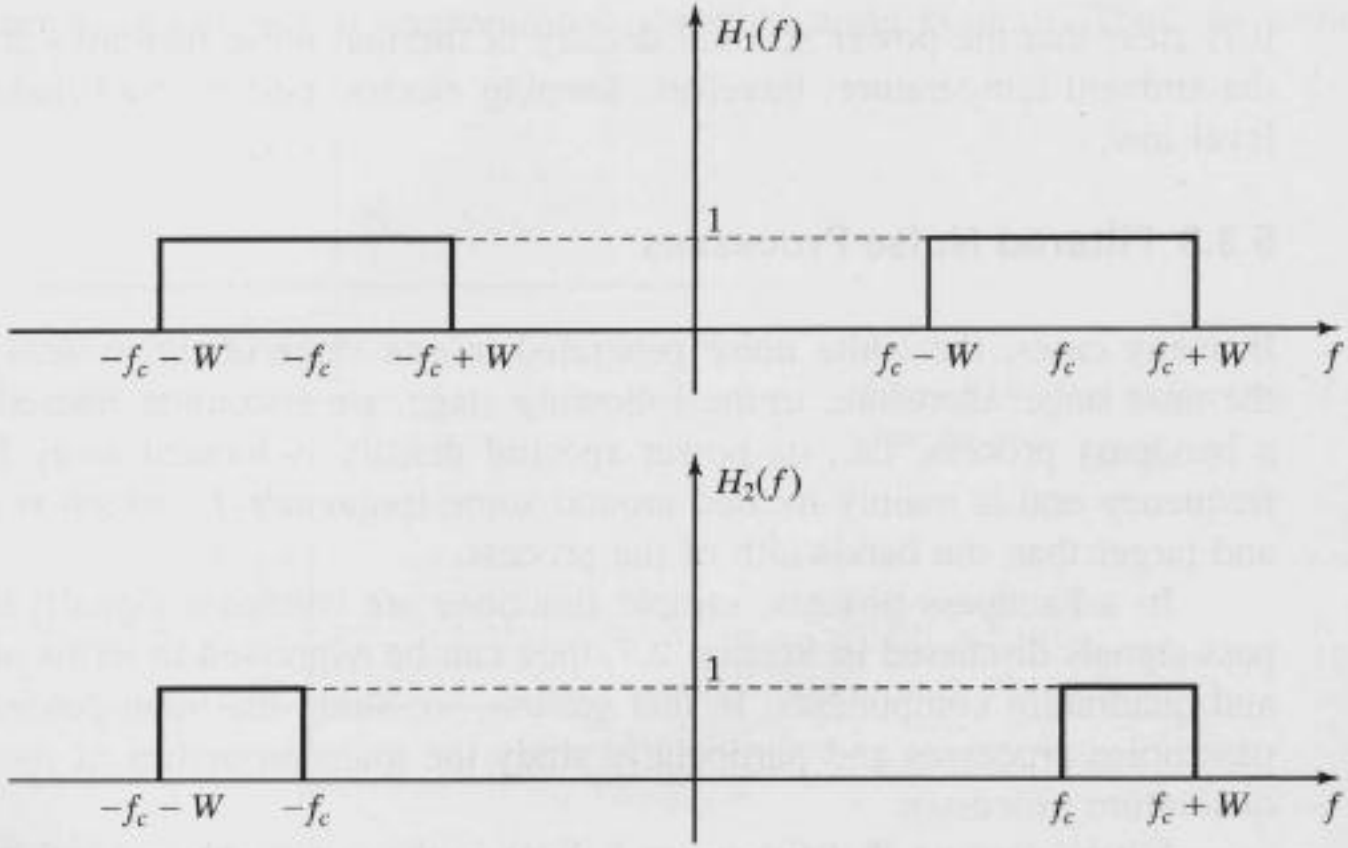

Figure 5.21 Filter transfer functions $H_1(f)$ and $H_2(f)$.

Filtered Noise Processes (4/5)

- The power spectral density of the filtered noise will be

$$S_X(f) = \frac{N_0}{2} |H(f)|^2 = \frac{N_0}{2} H(f)$$

where we have used the fact that for ideal filters $|H(f)|^2 = H(f)$

- For the two filtered noise processes, we have the following power spectral densities:

$$S_{X1}(f) = \begin{cases} \frac{N_0}{2}, & |f - f_c| \leq W \\ 0, & otherwise \end{cases}$$

and

$$S_{X2}(f) = \begin{cases} \frac{N_0}{2}, & f_c \leq |f| \leq f_c + W \\ 0, & otherwise \end{cases}$$

Filtered Noise Processes (5/5)

- All bandpass filtered noises have an in-phase and quadrature component that are lowpass signals
- The bandpass random process $X(t)$ can be expressed as

$$X(t) = X_c(t)\cos(2\pi f_c t) - X_s(t)\sin(2\pi f_c t)$$

where $X_c(t)$ and $X_s(t)$ are the in-phase and quadrature components, respectively

- $X_c(t)$ and $X_s(t)$ are lowpass processes
- We can represent the filtered noise in terms of its envelope and phase as

$$X(t) = A(t)\cos(2\pi f_c t + \Theta(f))$$

where $A(t)$ and $\Theta(f)$ are lowpass random process

Properties of the In-Phase and Quadrature Processes (1/4)

- For filtered white Gaussian noise, the following properties for $X_c(t)$ and $X_s(t)$ can be proved

1. $X_c(t)$ and $X_s(t)$ are zero-mean, lowpass, jointly WSS, and jointly Gaussian random processes
2. If the power in process $X(t)$ is P_X, then the power in each of the processes $X_c(t)$ and $X_s(t)$ is also P_X. In other words,
$$P_X = P_{X_c} = P_{X_s} = \int_{-\infty}^{\infty} S_X(f)df$$
3. Processes $X_c(t)$ and $X_s(t)$ have a common power spectral density. The power spectral densities can be either one shown in Fig. 5.22 when the white noise is filtered by $H_1(f)$ and $H_2(f)$, respectively

Properties of the In-Phase and Quadrature Processes (2/4)

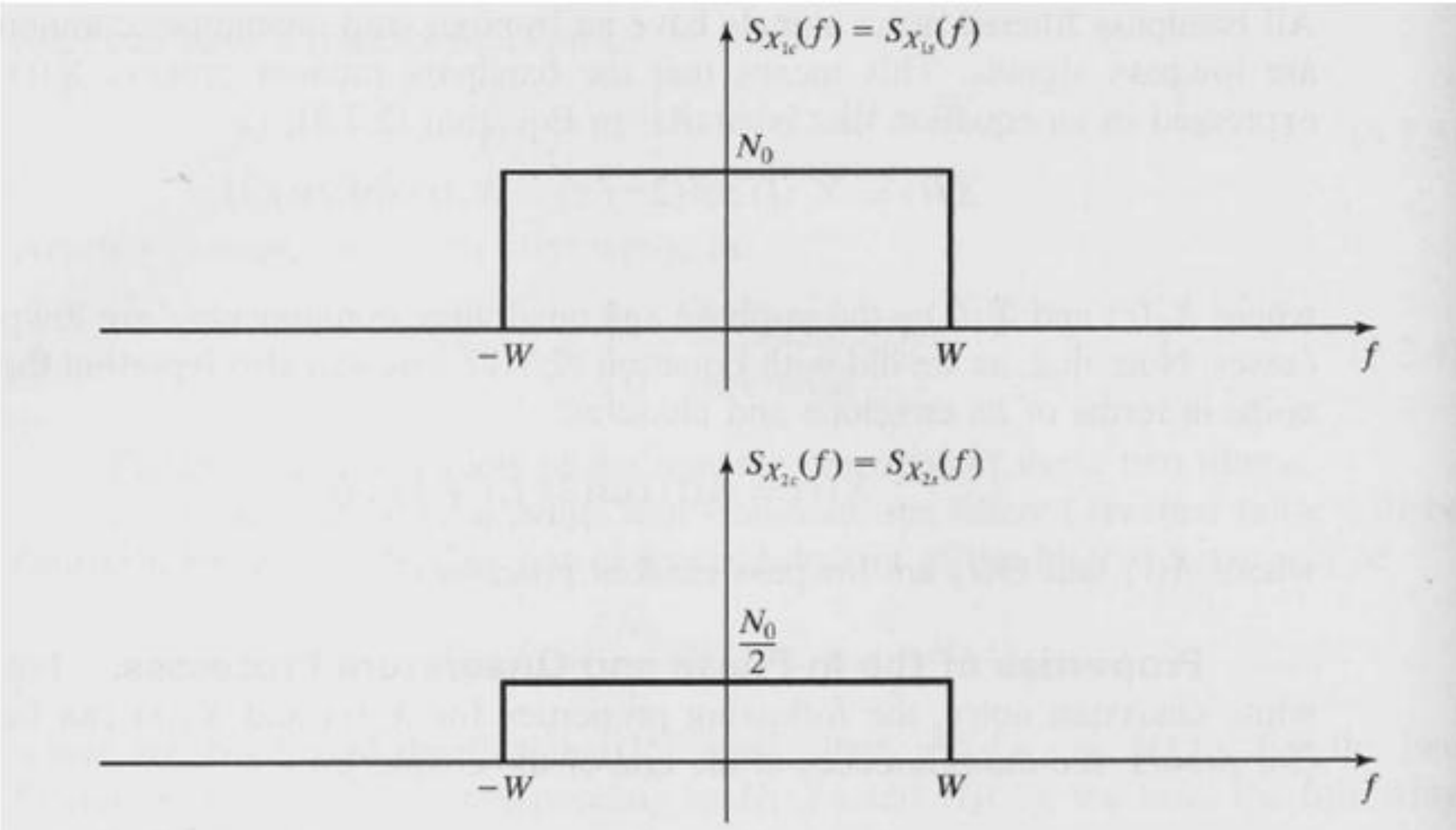

Figure 5.22 Power spectral densities of the in-phase and quadrature components of the filtered noise passed through $H_1(f)$ and $H_2(f)$.

Properties of the In-Phase and Quadrature Processes (3/4)

4. If $+f_c$ and $-f_c$ are the axis of symmetry of the positive and negative frequencies, then $X_c(t)$ and $X_s(t)$ will be independent processes. In other words, when the white noise is filtered by $H_1(f)$, then $X_c(t)$ and $X_s(t)$ are independent. It is not independent when filtered by $H_2(f)$

Properties of the In-Phase and Quadrature Processes (4/4)

- **Example 5.3.3.** For the bandpass white noise at the output of filter $H_1(f)$ as shown in Fig. 5.21, find the power spectral density of the process $Z(t)=aX_c(t)+bX_s(t)$
- Since f_c is the axis of symmetry of the noise power spectral density, the in-phase and quadrature components of the noise will be independent; therefore, we are dealing with the sum of two independent and zero-mean processes
- The power spectral density of $Z(t)$ is the sum of the power spectral densities of $aX_c(t)$ and $bX_s(t)$. Thus, $S_Z(f)=a^2S_{Xc}(f)+b^2S_{Xs}(f)$

Noise Equivalent Bandwidth (1/4)

- When a white Gaussian noise passes through a filter, the output process, will not be white anymore
- The filter characteristic shapes the spectral properties of the output process, we have

$$S_Y(f) = S_X(f)|H(f)|^2 = \frac{N_0}{2}|H(f)|^2$$

- The power content of the output process is

$$P_Y = \int_{-\infty}^{\infty} S_Y(f)df = \frac{N_0}{2}\int_{-\infty}^{\infty}|H(f)|^2\,df$$

- We define the noise equivalent bandwidth of a filter with the frequency response $H(f)$ as

$$B_{neq} = \frac{\int_{-\infty}^{\infty}|H(f)|^2\,df}{2H_{\max}^2}$$

Noise Equivalent Bandwidth (2/4)

- The power content of the output process can be written as

$$P_Y = \frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2\, df$$

$$= \frac{N_0}{2} \times 2B_{neq} H_{\max}^2$$

$$= N_0 B_{neq} H_{\max}^2$$

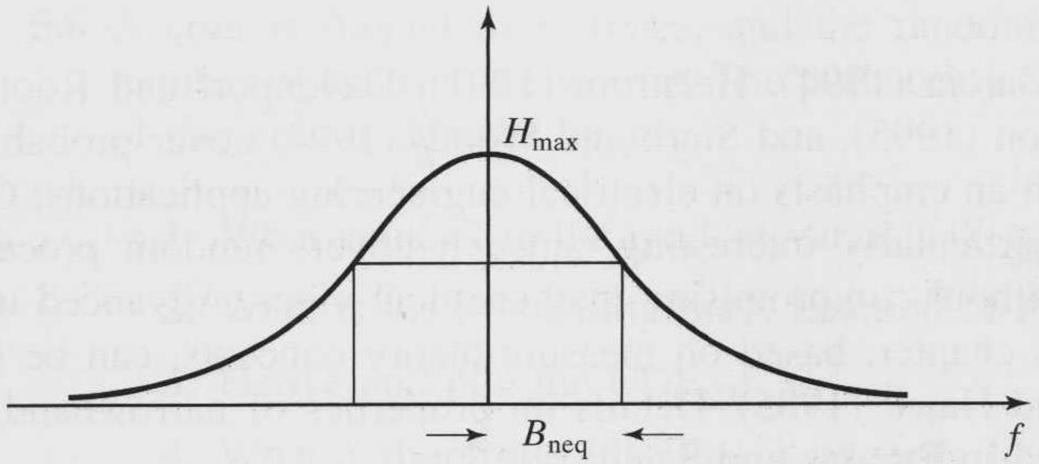

Figure 5.23 Noise equivalent bandwidth of a typical filter.

Noise Equivalent Bandwidth (3/4)

- **Example 5.3.4.** Find the noise equivalent bandwidth of a lowpass RC filter
- The transfer function of the lowpass RC filter is

$$H(f) = \frac{1}{1 + j2\pi fRC}$$

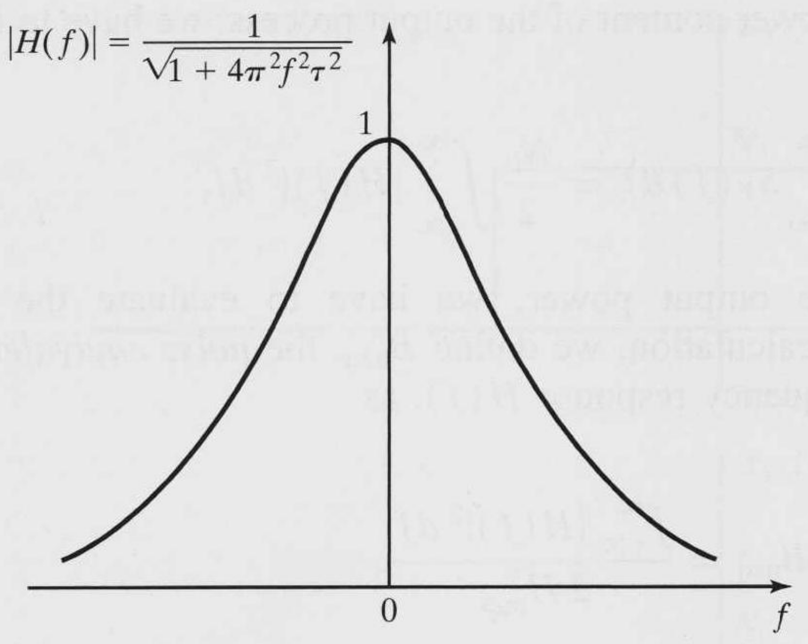

Figure 5.24 Frequency response of a lowpass *RC* filter.

Noise Equivalent Bandwidth (4/4)

- **Example 5.3.4. (Cont'd)** Defining $\tau = RC$, we have

$$|H(f)| = \frac{1}{\sqrt{1+4\pi^2 f^2 \tau^2}}$$

and therefore $H_{max} = 1$. We also have

$$\int_{-\infty}^{\infty} |H(f)|^2 \, df = 2\int_0^{\infty} \frac{1}{1+4\pi^2 f^2 \tau^2} df$$
$$= \frac{1}{2\tau}$$

Hence,

$$B_{neq} = \frac{\frac{1}{2\tau}}{2 \times 1} = \frac{1}{4RC}$$