Chapter 5 Probability and Random Processes (II)

Random Processes: Basic Concepts (1/6)

- The deterministic assumption on time-varying signals is not a valid assumption, and it is more appropriate to model signals as random rather than deterministic functions
- A *random process* is defined as the mapping of an element ω_i in the sample space Ω to a signal $x(t;\omega_i)$
- The realization of one from the set of possible signals is governed by some probabilistic law
- The difference between a random process and a random variables lies in that in random processes, we have signals (functions) instead of values (numbers)

Random Processes: Basic Concepts (2/6)

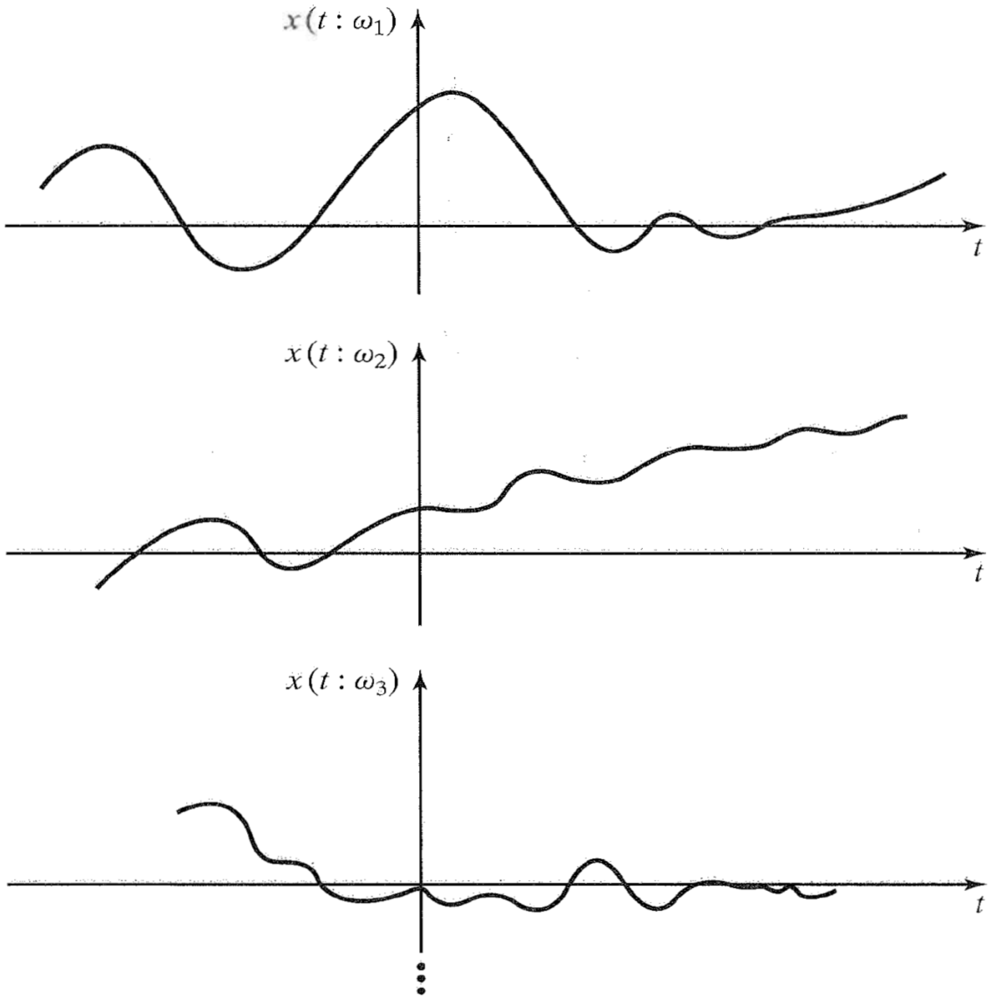

Figure 5.13 Sample functions of a random process.

Random Processes: Basic Concepts (3/6)

- For each ω_i, there exists a deterministic time function $x(t;\, \omega_i)$, which is called a *sample function* or a *realization* of the random process
- At each time instant t_0, the random process $X(t)$ is degenerated into a random variable denoted by $X(t_0)$. In other words, *at any time instant, the value of a random process is a random variable*

Random Processes: Basic Concepts (4/6)

- **Example 5.2.5.** Let Ω denote the sample space corresponding to the random experiment of throwing a dice. Obviously, in this case $\Omega=\{1,2,3,4,5,6\}$. For all ω_i, let $X(t;\omega_i)=\omega_i e^{-t}u_{-1}(t)$ denote a random process. Then $X(1)$ is a random variable taking value $e^{-1}, 2e^{-1}, \ldots, 6e^{-1}$ and each has probability 1/6. Sample functions of this random process are shown in Fig. 5.14.

Random Processes: Basic Concepts (5/6)

- **Example 5.2.5. (Cont'd)**

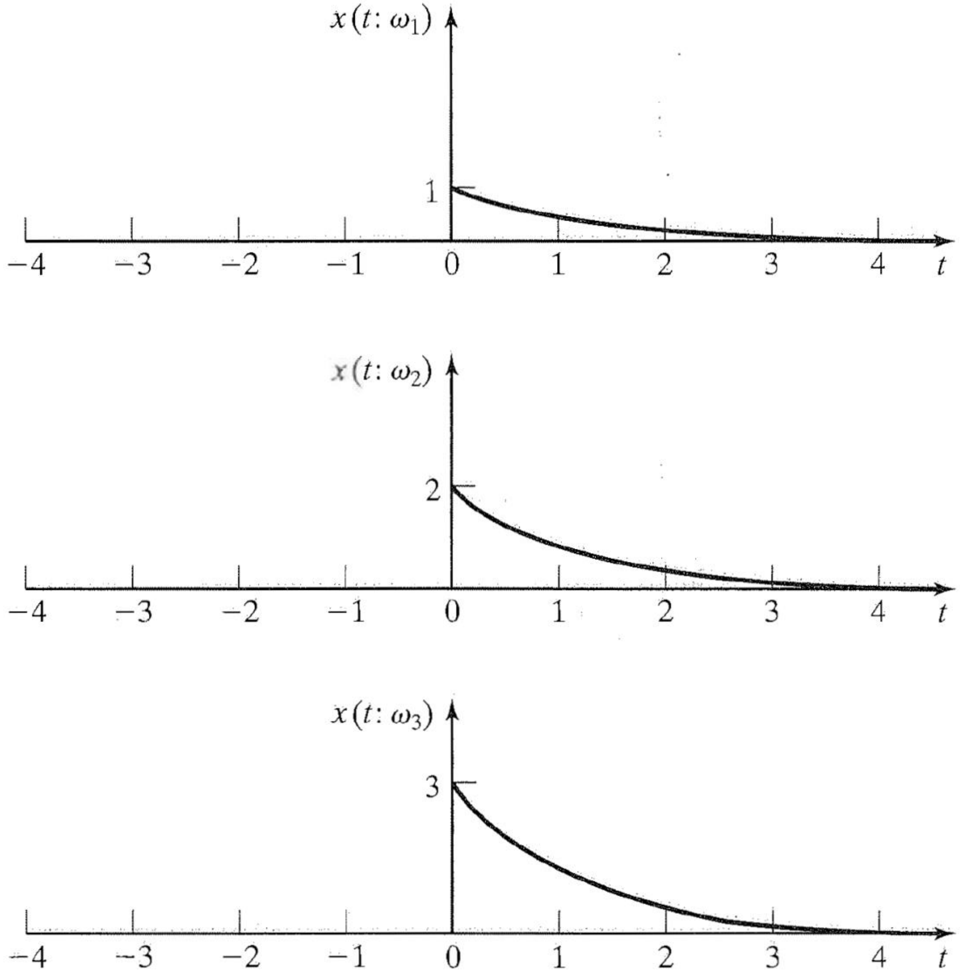

Figure 5.14 Sample functions of Example 5.2.5.

Random Processes: Basic Concepts (6/6)

- **Example 5.2.6.** We can have discrete-time random processes, which are similar to continuous-time random processes. For instance at any time instant let X_i denote the outcome of a random experiment consisting of independent drawings from a Gaussian random variable distributed according to $\mathcal{N}(0,1)$.

Statistical Averages (1/6)

- Define the *mean*, or *expectation*, of the random process $X(t)$ as a deterministic function of time by $m_X(t)$ that at each time instant t_0 equals the mean of the random variable $X(t_0)$. That is, $m_X(t)=E[X(t)]$ for all t
- At any t_0, the random variable $X(t_0)$ is well defined with a probability density function $f_{X(t_0)}(x)$, we have

$$E[X(t_0)] = m_X(t_0) = \int_{-\infty}^{\infty} x f_{X(t_0)}(x)dx$$

- Fig. 5.15 gives a pictorial description of this definition

Statistical Averages (2/6)

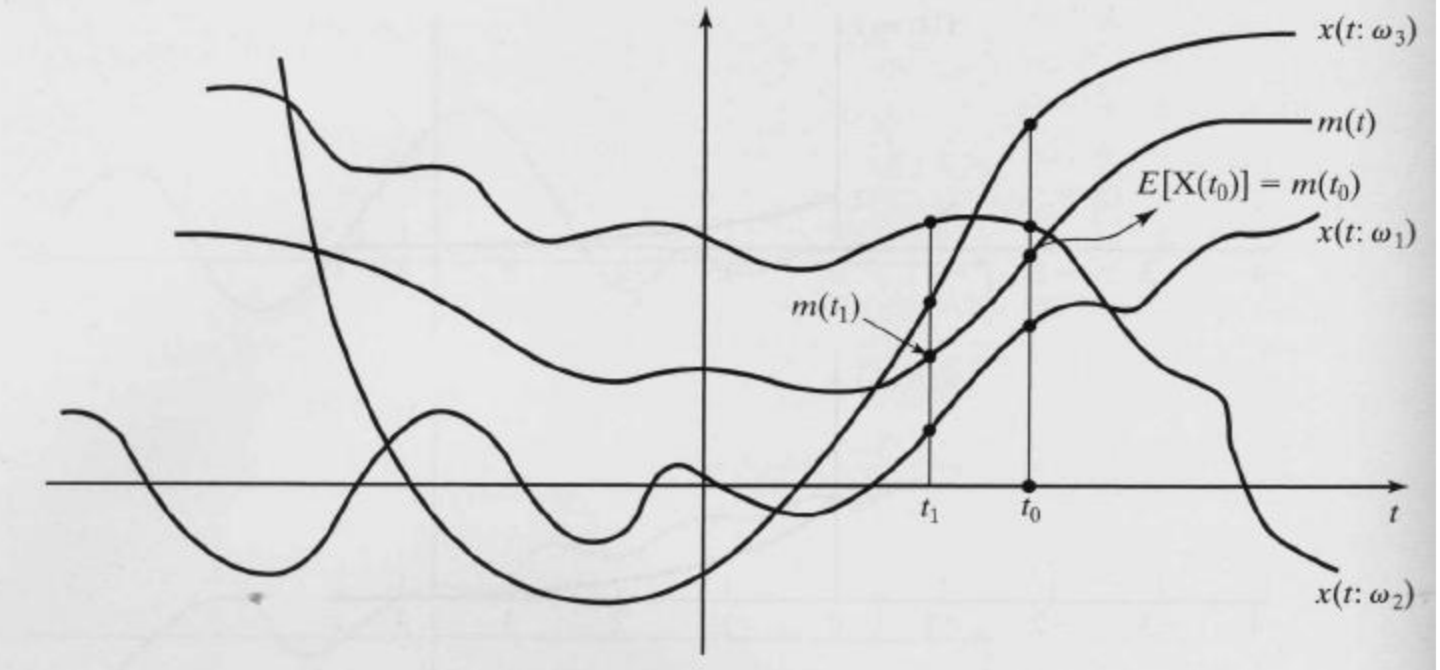

Figure 5.15 The mean of a random process.

Statistical Averages (3/6)

- **Example 5.2.7.** Let the random process $X(t)$=$A\cos(2\pi f_0 t+\Theta)$, where A and f_0 denote the fixed amplitude and frequency and Θ denotes the random phase. Θ is uniformly distributed on $[0, 2\pi)$. Please find $E[X(t)]$
- We have

$$f_\Theta(\theta)=\begin{cases}\frac{1}{2\pi}, & 0\le\theta<2\pi \\ 0, & otherwise\end{cases}.$$

Hence,

$$E[X(t)]=\int_0^{2\pi} A\cos(2\pi f_0 t+\theta)\frac{1}{2\pi}d\theta=0$$

We observe that $m_X(t)$ is independent of t

Statistical Averages (4/6)

- The *autocorrelation function* of the random process $X(t)$, denoted by $R_X(t_1,t_2)$, is defined by $R_X(t_1,t_2)=E[X(t_1)X(t_2)]$
- $R_X(t_1,t_2)$ is a deterministic function of two variables t_1 and t_2 given by

$$R_X(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{X(t_1),X(t_2)}(x_1,x_2)dx_1 dx_2$$

- The autocorrelation function is important because it completely describes the power spectral density and the power content of a large class of random processes

Statistical Averages (5/6)

- **Example 5.2.8.** The random process $X(t)$ is defined in Example 5.2.7. Please find the autocorrelation function of $X(t)$.
- We have

$$R_X(t_1,t_2) = E[A\cos(2\pi f_0 t_1 + \Theta)A\cos(2\pi f_0 t_2 + \Theta)]$$

$$= A^2 E[\tfrac{1}{2}\cos(2\pi f_0(t_1 - t_2)) + \tfrac{1}{2}\cos(2\pi f_0(t_1 + t_2) + 2\Theta)]$$

$$= \tfrac{A^2}{2}\cos(2\pi f_0(t_1 - t_2)),$$

where we have used

$$E[\cos(2\pi f_0(t_1 + t_2) + 2\Theta)] = \int_0^{2\pi} \cos(2\pi f_0(t_1 + t_2) + 2\theta)\tfrac{1}{2\pi}\,d\theta = 0$$

Statistical Averages (6/6)

- **Example 5.2.9.** The process $X(t)=X$, where X is a random variable uniformly distributed on [-1,1]. Find the autocorrelation function of $X(t)$.
- $R_X(t_1,t_2)=E(X^2)=\int_{-1}^{+1}\frac{x^2}{2}dx=\frac{1}{3}$

Wide-Sense Stationary Processes (1/2)

- A process $X(t)$ is wide-sense stationary (WSS) if the following conditions are satisfied:
  - $m_X(t)=E(X(t))$ is independent of t
  - $R_X(t_1,t_2)$ depends only on the time difference $\tau=t_1-t_2$ and not on t_1 and t_2 individually
- For WSS processes, their mean and autocorrelation will be denoted by m_X and $R_X(\tau)$
- **Example 5.2.10.** The random process $X(t)$ is defined as $X(t)=A\cos(2\pi f_0 t+\Theta)$, where A and f_0 denote amplitude and frequency. Θ is uniformly distributed on $[0,2\pi)$.
- $E[X(\mathrm{t})]=0$, $R_X(t_1,t_2)=A^2/2\cos(2\pi f_0(t_1-t_2))$, $X(t)$ is WSS

Wide-Sense Stationary Processes (2/2)

- **Example 5.2.11.** Let the random process $Y(t)$ be similar to the random $X(t)$ in Example 5.2.10, but assume that Θ is uniformly distributed between 0 and π
- We have

$$m_Y(t) = E[A\cos(2\pi f_0 t + \Theta)]$$
$$= A\int_0^{\pi} \tfrac{1}{\pi}\cos(2\pi f_0 t + \theta)d\theta$$
$$= -\tfrac{2A}{\pi}\sin(2\pi f_0 t)$$

$m_Y(t)$ is not independent of t, the process $Y(t)$ is not WSS

- From the definition of autocorrelation function, it follows that $R_X(t_1,t_2)=R_X(t_2,t_1)$. If $X(t)$ is WSS, we have $R_X(\tau)=R_X(-\tau)$, i.e., the autocorrelation function is an even function in WSS processes

Multiple Random Processes (1/2)

- Two random processes $X(t)$ and $Y(t)$ are *independent* if, for all t_1, t_2, the random variables $X(t_1)$ and $Y(t_2)$ are independent. Similarly, $X(t)$ and $Y(t)$ are *uncorrelated* if $X(t_1)$ and $Y(t_2)$ are uncorrelated for all t_1, t_2
- The *cross correlation* between two random processes $X(t)$ and $Y(t)$ is defined as

$$R_{XY}(t_1,t_2)=E[X(t_1)Y(t_2)]$$

In other words, we have

$$R_{XY}(t_1,t_2)=R_{YX}(t_2,t_1)$$

Multiple Random Processes (2/2)

- Two random processes $X(t)$ and $Y(t)$ are jointly WSS, if both $X(t)$ and $Y(t)$ are individually WSS and the cross correlation $R_{XY}(t_1,t_2)$ depends only on $\tau=t_1-t_2$
- Note that for jointly WSS random processes, from the definition and the relation $R_{XY}(t_1,t_2)=R_{YX}(t_2,t_1)$, it follows that

$$R_{XY}(\tau) = R_{YX}(-\tau)$$

- **Example 5.2.12.** Assume that the two random processes $X(t)$ and $Y(t)$ are jointly WSS, determine the autocorrelation of the process $Z(t)=X(t)+Y(t)$
- By definition,

$$\begin{aligned} R_Z(t+\tau,t) &= E[Z(t+\tau)Z(t)] \\ &= E[(X(t+\tau)+Y(t+\tau))(X(t)+Y(t))] \\ &= R_X(\tau)+R_Y(\tau)+R_{XY}(\tau)+R_{XY}(-\tau) \end{aligned}$$

Random Processes and Linear Systems (1/7)

- When a random process passes through a linear time-invariant system, the output is also a random process
- We assume that a WSS process $X(t)$ is the input to a linear time-invariant system with the impulse response $h(t)$ and the output process is denoted by $Y(t)$

Random Processes and Linear Systems (2/7)

- We will demonstrate that if a WSS process $X(t)$ with mean m_X and autocorrelation function $R_X(\tau)$ is passed through a linear time-invariant system with impulse response $h(t)$, the input and output process $X(t)$ and $Y(t)$ will be jointly WSS with

$$m_Y = m_X \int_{-\infty}^{\infty} h(t)dt$$

$$R_{XY}(\tau) = R_X(\tau) * h(-\tau)$$

$$\begin{aligned} R_Y(\tau) &= R_X(\tau) * h(\tau) * h(-\tau) \\ &= R_X(\tau) * h(-\tau) * h(\tau) \\ &= R_{XY}(\tau) * h(\tau) \end{aligned}$$

Random Processes and Linear Systems (3/7)

- By using the convolution integral to relate the output $Y(t)$ to the input $X(t)$, we have

$$E[Y(t)] = E\left[\int_{-\infty}^{\infty} X(\tau)h(t-\tau)d\tau\right]$$

$$= \int_{-\infty}^{\infty} E[X(\tau)]h(t-\tau)d\tau$$

$$= \int_{-\infty}^{\infty} m_X h(t-\tau)d\tau$$

$$\overset{u=t-\tau}{=} m_X \int_{-\infty}^{\infty} h(u)du \equiv m_Y$$

- This proves that m_Y is independent of t

Random Processes and Linear Systems (4/7)

- The cross correlation function between the output and the input is

$$E[X(t_1)Y(t_2)] = E\left[X(t_1)\int_{-\infty}^{\infty} X(s)h(t_2-s)ds\right]$$
$$= \int_{-\infty}^{\infty} E[X(t_1)X(s)]h(t_2-s)ds$$
$$= \int_{-\infty}^{\infty} R_X(t_1-s)h(t_2-s)ds$$
$$\overset{u=s-t_2}{=} \int_{-\infty}^{\infty} R_X(t_1-t_2-u)h(-u)du$$
$$= \int_{-\infty}^{\infty} R_X(\tau-u)h(-u)du$$
$$= R_X(\tau) * h(-\tau) \equiv R_{XY}(\tau)$$

- $R_{XY}(t_1,t_2)$ depends only on $\tau = t_1 - t_2$

Random Processes and Linear Systems (5/7)

- The autocorrelation function of the output is

$$E[Y(t_1)Y(t_2)] = E\left[\left(\int_{-\infty}^{\infty} X(s)h(t_1-s)ds\right)Y(t_2)\right]$$
$$= \int_{-\infty}^{\infty} R_{XY}(s-t_2)h(t_1-s)ds$$
$$\overset{u=s-t_2}{=} \int_{-\infty}^{\infty} R_{XY}(u)h(t_1-t_2-u)du$$
$$= R_{XY}(\tau) * h(\tau)$$
$$= R_X(\tau) * h(-\tau) * h(\tau)$$

- $R_Y(t_1,t_2)$ and $R_{XY}(t_1,t_2)$ depend only on $\tau = t_1 - t_2$. Thus, the output process is WSS. The input and output processes are jointly WSS

Random Processes and Linear Systems (6/7)

- **Example 5.2.13.** Assume a WSS process passes through a differentiator. What are the mean and autocorrelation functions of the output? What is the cross correlation between the input and output?
- In a differentiator, $h(t)=\delta'(t)$. Since $\delta'(t)$ is odd, it follows that

$$m_Y = m_X \int_{-\infty}^{\infty} \delta'(t)dt = 0$$

The cross correlation function between output and input is

$$R_{XY}(\tau) = R_X(\tau) * \delta'(-\tau) = -R_X(\tau) * \delta'(\tau) = -\tfrac{d}{d\tau} R_X(\tau)$$

and the autocorrelation function of the output is

$$R_Y(\tau) = -\tfrac{d}{d\tau} R_X(\tau) * \delta'(\tau) = -\tfrac{d^2}{d\tau^2} R_X(\tau)$$

Random Processes and Linear Systems (7/7)

- **Example 5.2.14.** Repeat Example 5.2.13 for the case where the LTI system is a quadrature filter defined by $h(t) = \frac{1}{\pi t}$; therefore, $H(f)$=-$j$sgn(f). In this case, the output of the filter is the Hilbert transform of the input
- We have

$$m_Y = m_X \int_{-\infty}^{\infty} \frac{1}{\pi t} dt = 0$$

The cross correlation function is

$$R_{XY}(\tau) = R_X(\tau) * \frac{1}{-\pi t} = -\hat{R}_X(\tau)$$

and the autocorrelation function of the output is

$$R_Y(\tau) = R_{XY}(\tau) * \frac{1}{\pi t} = -\hat{\hat{R}}_X(\tau) = R_X(\tau)$$

where we assume that the $R_X(\tau)$ has no DC component

Power Spectral Density of WSS Processes (1/6)

- If the signals of the random process are slowly varying, then the random process will mainly contain low frequencies and its power will be mostly concentrated at low frequencies
- If the signals change very fast, then most of the power in the random process will be at the high frequency components
- A useful function that determines the distribution of the power of the random process at different frequencies is the *power spectral density* or *power spectrum* of the random process
- The power spectral density of a random process $X(t)$ is denoted by $S_X(f)$, and denotes the strength of the power in the random process as a function of frequency
- The unit of $S_X(f)$ is Watts/Hz

Power Spectral Density of WSS Processes (2/6)

- For WSS processes, the *Wiener-Khinchin* theorem relates the power spectrum of the random process to its autocorrelation function
- **Wiener-Khinchin Theorem.** For a WSS random process $X(t)$, the power spectral density is the Fourier transform of the autocorrelation function, i.e.,

$$S_X(f) = \mathcal{F}[R_X(\tau)]$$

- **Example 5.2.15.** For the WSS random process $X(t) = A\cos(2\pi f_0 t + \Theta)$, we have

$$R_X(\tau) = \tfrac{A^2}{2}\cos(2\pi f_0 \tau)$$

Hence,

$$S_X(f) = \tfrac{A^2}{4}[\delta(f - f_0) + \delta(f + f_0)]$$

Power Spectral Density of WSS Processes (3/6)

- **Example 5.2.15. (Cont'd)** All the power content of the process is located at f_0 and $-f_0$ because the sample functions of this process are sinusoids with their power at those frequencies

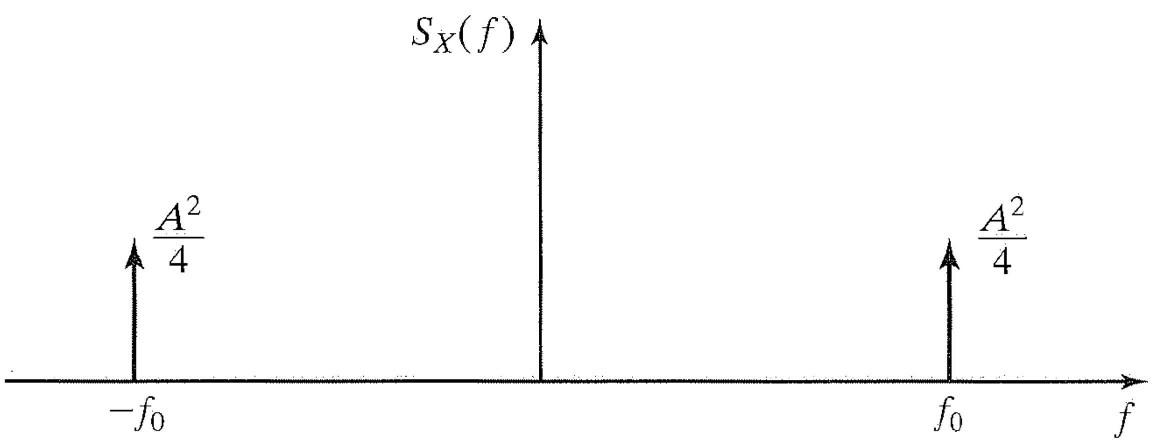

Figure 5.17 Power spectral density of the random process of Example 5.2.15.

Power Spectral Density of WSS Processes (4/6)

- The power content, or simply the *power*, of a random process is the sum of powers at all frequencies in that random process. This means

$$P_X = \int_{-\infty}^{\infty} S_X(f)df$$

- Since $S_X(f)$ is the Fourier transform of $R_X(\tau)$, then $R_X(\tau)$ will be the inverse Fourier of $S_X(f)$. We have

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f)e^{j2\pi f\tau}df.$$

Substituting $\tau=0$ into this relation yields

$$R_X(0) = \int_{-\infty}^{\infty} S_X(f)df.$$

We conclude that

$$P_X=R_X(0)$$

Power Spectral Density of WSS Processes (5/6)

- The power in a WSS random process can be found either by integrating its power spectral density or substituting $\tau=0$ in the autocorrelation function of the process
- **Example 5.2.17.** Find the power in the process given in Example 5.2.15.
- Notice that

$$R_X(\tau) = \frac{A^2}{2}\cos(2\pi f_0 \tau)$$

and

$$S_X(f) = \frac{A^2}{4}[\delta(f - f_0) + \delta(f + f_0)].$$

Power Spectral Density of WSS Processes (6/6)

- **Example 5.2.17. (Cont'd)** We can use either the relation

$$
\begin{aligned}
P_X &= \int_{-\infty}^{\infty} S_X(f)\,df \\
&= \int_{-\infty}^{\infty} \left[\frac{A^2}{4}[\delta(f - f_0) + \delta(f + f_0)]\right] df \\
&= 2 \times \frac{A^2}{4} \\
&= \frac{A^2}{2}
\end{aligned}
$$

or the relation

$$
\begin{aligned}
P_X &= R_X(0) \\
&= \frac{A^2}{2}\cos(2\pi f_0 \tau)\Big|_{\tau=0} \\
&= \frac{A^2}{2}
\end{aligned}
$$

Power Spectra in LTI Systems (1/5)

- We have proved that when a WSS process with mean m_X and autocorrelation $R_X(\tau)$ passes through a LTI system with the impulse response $h(t)$, the output process will be also WSS with mean

$$m_Y = m_X \int_{-\infty}^{\infty} h(t)dt$$

and autocorrelation

$$R_Y(\tau) = R_X(\tau) * h(\tau) * h(-\tau).$$

- Note $X(t)$ and $Y(t)$ will be jointly WSS with the cross-correlation function

$$R_{XY}(\tau) = R_X(\tau) * h(-\tau)$$

Power Spectra in LTI Systems (2/5)

- We can compute the Fourier transform of both sides of these relations to obtain

$$m_Y = m_X H(0)$$

$$S_Y(f) = S_X(f)\,|H(f)|^2$$

- Note that we already use $\mathscr{F}[h(-\tau)] = H^*(f)$
- The first equation says that the mean value of the response of the system only depends on the value of $H(f)$ at f=0 (DC response)
- The second equation says that when dealing with the power spectrum, the phase of $H(f)$ is irrelevant; only the magnitude of $H(f)$ affects the output power spectrum

Power Spectra in LTI Systems (3/5)

- If a random process passes through a differentiator, we have $H(f)=j2\pi f$; hence,

$$m_Y = m_X H(0) = 0$$

$$S_Y(f) = 4\pi^2 f^2 S_X(f)$$

- Let us define the *cross spectral density* $S_{XY}(f)$ as

$$S_{XY}(f)=\mathscr{F}[R_{XY}(\tau)].$$

Then

$$S_{XY}(f)=S_X(f)H^*(f),$$

and since $R_{YX}(\tau)=R_{XY}(-\tau)$, we have

$$S_{YX}(f)=S^*_{XY}(f)=S_X(f)H(f)$$

(Hint: $R_X(\tau)$ is real and even, $\mathscr{F}[R_X(\tau)]=S_X(f)$ is real and even.)

Power Spectra in LTI Systems (4/5)

- Note that $S_X(f)$ and $S_Y(f)$ are real non-negative functions, $S_{XY}(f)$ and $S_{YX}(f)$ can generally be complex functions

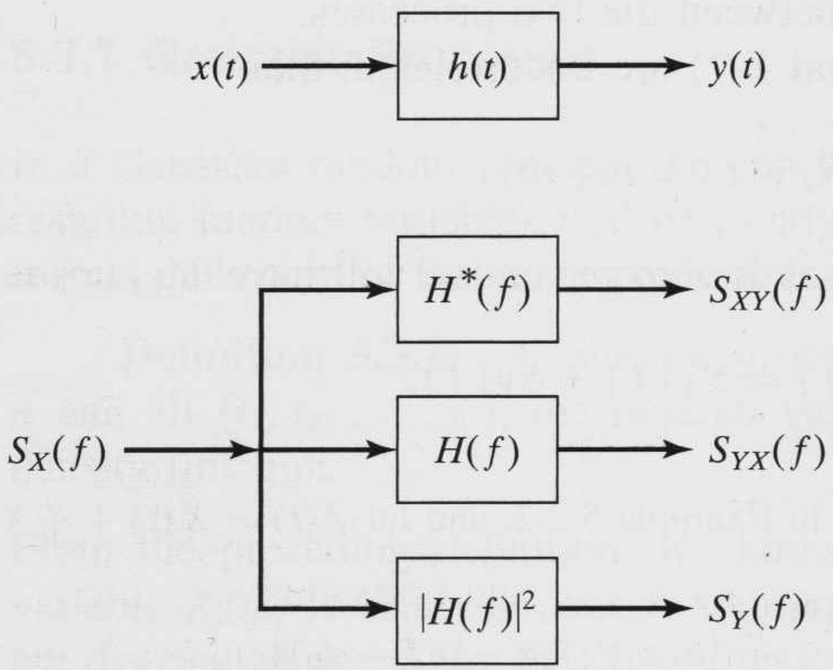

Figure 5.18 Input–output relations for the power spectral density and the cross spectral density.

Power Spectra in LTI Systems (5/5)

- **Example 5.2.18.** If the process in Example 5.2.2 passes through a differentiator, we have $H(f)=j2\pi f$. Since $X(t)=A\cos(2\pi f_0 t+\Theta)$, Θ is uniformly distributed between 0 and 2π. Then,

$$S_X(f)=\tfrac{A^2}{4}[\delta(f-f_0)+\delta(f+f_0)].$$

- Therefore,

$$S_Y(f)=4\pi^2 f^2\left[\tfrac{A^2}{4}[\delta(f-f_0)+\delta(f+f_0)]\right]$$

and

$$S_{XY}(f)=(-j2\pi f)S_X(f)=-\tfrac{jA^2\pi f}{2}\left[\delta(f-f_0)+\delta(f+f_0)\right]$$

Power Spectral Density of a Sum Process (1/4)

- Let $Z(t)=X(t)+Y(t)$, where $X(t)$ and $Y(t)$ are jointly WSS processes
- We already know that $Z(t)$ is a WSS process with

$$R_Z(\tau)=R_X(\tau)+R_Y(\tau)+R_{XY}(\tau)+R_{YX}(\tau) \qquad (5.2.21)$$

- We have $R_{XY}(\tau)=R_{YX}(-\tau)$. From this information, conclude that $S_{XY}(f)=S^*_{YX}(f)$
- Taking the Fourier transform of both sides of Eq. (5.2.21), we have

$$\begin{aligned} S_Z(f) &= S_X(f)+S_Y(f)+S_{XY}(f)+S_{YX}(f) \\ &= S_X(f)+S_Y(f)+S_{XY}(f)+S_{XY}^*(f) \\ &= S_X(f)+S_Y(f)+2\mathrm{Re}[S_{XY}(f)] \end{aligned}$$

Power Spectral Density of a Sum Process (2/4)

- The power spectral density of the sum process is the sum of the power spectra of the individual processes plus a third term, which depends on the cross correlation between the two processes
- If two WSS processes $X(t+\tau)$ and $Y(t)$ are uncorrelated, then we have

$$\begin{aligned}&\text{COV}(X(t+\tau),Y(t))\\&=E[X(t+\tau)Y(t)]-E[X(t+\tau)]E[Y(t)]\\&=0\end{aligned}$$

Thus, we have

$$\begin{aligned}E[X(t+\tau)Y(t)]&=E[X(t+\tau)]E[Y(t)]\\&=m_X m_Y\end{aligned}$$

Power Spectral Density of a Sum Process (3/4)

- If at least one of the processes is zero mean, we will have $R_{XY}(\tau)=0$ and

$$S_Z(f)=S_X(f)+S_Y(f)$$

- **Example 5.2.20.** Let the random process $X(t)$ is defined as $X(t)=A\cos(2\pi f_0 t+\Theta)$, where A and f_0 denote the fixed amplitude and frequency and Θ denotes the random phase. Θ is uniformly distributed between 0 and 2π. Let $Z(t)=X(t)+\frac{d}{dt}X(t)$. Please find $S_Z(f)$.
- Let $Y(t)=dX(t)/dt$ and $h(t)$ be the impulse response of a differentiator. $Y(t)$ and $X(t)$ are the output and input processes of the differentiator, respectively

Power Spectral Density of a Sum Process (4/4)

- **Example 5.2.20. (Cont'd)** From previous examples, we know that $H(f) = j2\pi f$ and $S_X(f) = \frac{A^2}{4}[\delta(f - f_0) + \delta(f + f_0)]$. Then,

$$S_{XY}(f) = S_X(f) H^*(f)$$
$$= -j\frac{A^2\pi f}{2}[\delta(f - f_0) + \delta(f + f_0)];$$

therefore,

$$\mathrm{Re}[S_{XY}(f)] = 0$$

From Example 5.2.18, we also know that

$$S_Y(f) = A^2\pi^2 f_0^{\ 2}[\delta(f - f_0) + \delta(f + f_0)].$$

- Hence

$$S_Z(f) = S_X(f) + S_Y(f)$$
$$= A^2(\tfrac{1}{4} + \pi^2 f_0^{\ 2})[\delta(f - f_0) + \delta(f + f_0)]$$