Chapter 4 Angle Modulation (I)

Angle Modulation (1/2)

- In frequency modulation (FM) systems, the frequency of the carrier f_c is changed by the message signal
- In phase modulation (PM) systems, the phase of the carrier is changed according to the variations in the message signal
- Frequency and phase modulation are nonlinear, and often they are jointly called *angle-modulation methods*
- Frequency and phase modulation systems generally expand the bandwidth such that the effective bandwidth of the modulated signal is usually many times the bandwidth of the message signal

Angle Modulation (2/2)

- The major benefit of these systems is their high degree of noise immunity. In fact, these systems sacrifice bandwidth for high-noise immunity
- Another advantage of angle-modulated signals is their constant envelope, which is beneficial when amplified by nonlinear amplifiers

Representation of FM and PM Signals (1/7)

- An angle-modulated signal generally can be written as

$$u(t) = A_c \cos(2\pi f_c t + \phi(t)),$$

where f_c denotes the carrier frequency and $\phi(t)$ denotes a time-varying phase

- The instantaneous frequency of this signal is given by

$$f_i(t) = f_c + \frac{1}{2\pi}\frac{d}{dt}\phi(t)$$

- If $m(t)$ is the message signal, then in a PM system, the phase is proportional to the message, i.e.,

$$\phi(t) = k_p m(t),$$

and in an FM system,

$$f_i(t) - f_c = k_f m(t)$$

Representation of FM and PM Signals (2/7)

- k_p and k_f are phase and frequency deviation constants. We have

$$\phi(t) = \begin{cases} k_p m(t), & PM \\ 2\pi k_f \int_{-\infty}^{t} m(\tau) d\tau, & FM \end{cases}.$$

On the other hand, this relation can be expressed as

$$\frac{d}{dt}\phi(t) = \begin{cases} k_p \frac{d}{dt} m(t), & PM \\ 2\pi k_f m(t), & FM \end{cases}.$$

- The demodulation of an FM signal involves finding the instantaneous frequency of the modulated signal and subtracting the carrier frequency from it
- In demodulation of PM, the demodulation process is done by finding the phase of the signal and then recovering $m(t)$

Representation of FM and PM Signals (3/7)

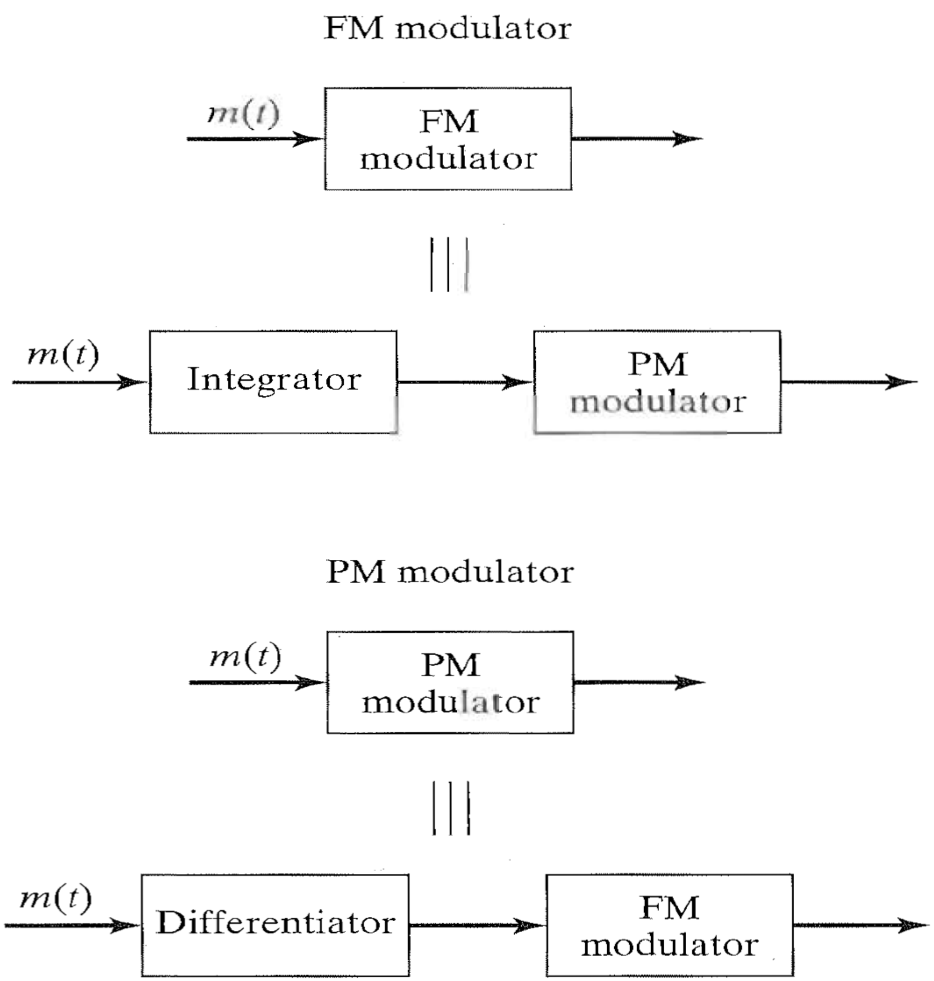


Figure 4.1 A comparison of frequency and phase modulators.

Representation of FM and PM Signals (4/7)

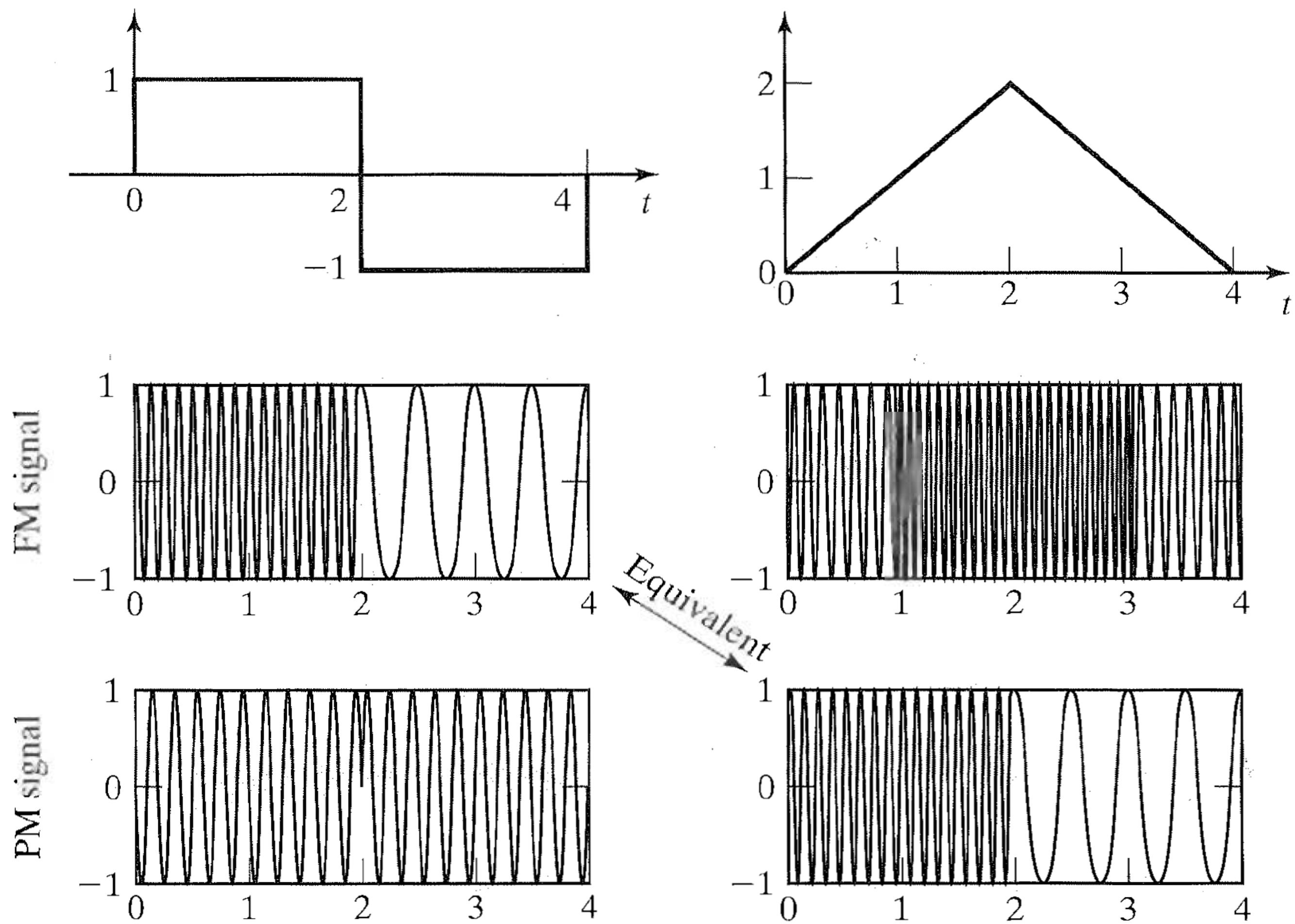


Figure 4.2 Frequency and phase modulation of square and sawtooth waves.

Representation of FM and PM Signals (5/7)

- The maximum phase deviation in a PM system is

$$\Delta\phi_{\max} = k_p \max[|m(t)|]$$

- The maximum frequency deviation in an FM system is

$$\Delta f_{\max} = k_f \max[|m(t)|]$$

- **Example 4.1.1.** The message signal $m(t)=a\cos(2\pi f_m t)$ is used to either frequency modulate or phase modulate the carrier $A_c\cos(2\pi f_c t)$. Find the modulated signal in each case.
- In PM, we have

$$\phi(t) = k_p m(t) = k_p a\cos(2\pi f_m t),$$

and in FM, we have

$$\phi(t) = 2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau = \frac{k_f a}{f_m}\sin(2\pi f_m t)$$

Representation of FM and PM Signals (6/7)

- **Example 4.1.1. (Cont'd)** The modulated signals will be

$$u(t) = \begin{cases} A_c \cos(2\pi f_c t + k_p a \cos(2\pi f_m t)), & PM \\ A_c \cos(2\pi f_c t + \frac{k_f a}{f_m} \sin(2\pi f_m t)), & FM \end{cases}.$$

- Define

$$\beta_p = k_p a$$

and

$$\beta_f = \frac{k_f a}{f_m}$$

- β_p: modulation indices of the PM systems
- β_f: modulation indices of the FM systems

Representation of FM and PM Signals (7/7)

- We can extend the definition of the modulation index for a general nonsinusoidal signal $m(t)$ as

$$\beta_p = k_p \max[|m(t)|]$$

$$\beta_f = \frac{k_f \max[|m(t)|]}{W},$$

where W denotes the bandwidth of the message signal $m(t)$

- In terms of the maximum phase and frequency deviation $\Delta\phi_{\max}$ and $\Delta f_{\max}$, we have

$$\beta_p = \Delta\phi_{\max};$$

$$\beta_f = \frac{\Delta f_{\max}}{W}$$

Narrowband Angle Modulation (1/3)

- Also known as *low-index angle modulation*
- Consider an angle modulation system in which the deviation constants k_p and k_f and the message signal $m(t)$ are such that for all t, we have $\phi(t) << 1$
- $u(t) = A_c \cos(2\pi f_c t + \phi(t))$ can be approximated as

$$u(t) = A_c \cos(2\pi f_c t)\cos\phi(t) - A_c \sin(2\pi f_c t)\sin\phi(t)$$

$$\cong A_c \cos(2\pi f_c t) - A_c \phi(t)\sin(2\pi f_c t)$$

- The modulated signal is very similar to a conventional-AM signal. The only difference is the message signal $m(t)$ is modulated on a sine carrier rather than a cosine carrier

Narrowband Angle Modulation (2/3)

- The bandwidth occupancy of narrowband angle modulation is similar to conventional AM modulation
- The narrowband angle-modulation method does not provide better noise immunity than a conventional AM system. Therefore, narrowband angle-modulation is seldom used in practice for communication purposes

Narrowband Angle Modulation (3/3)

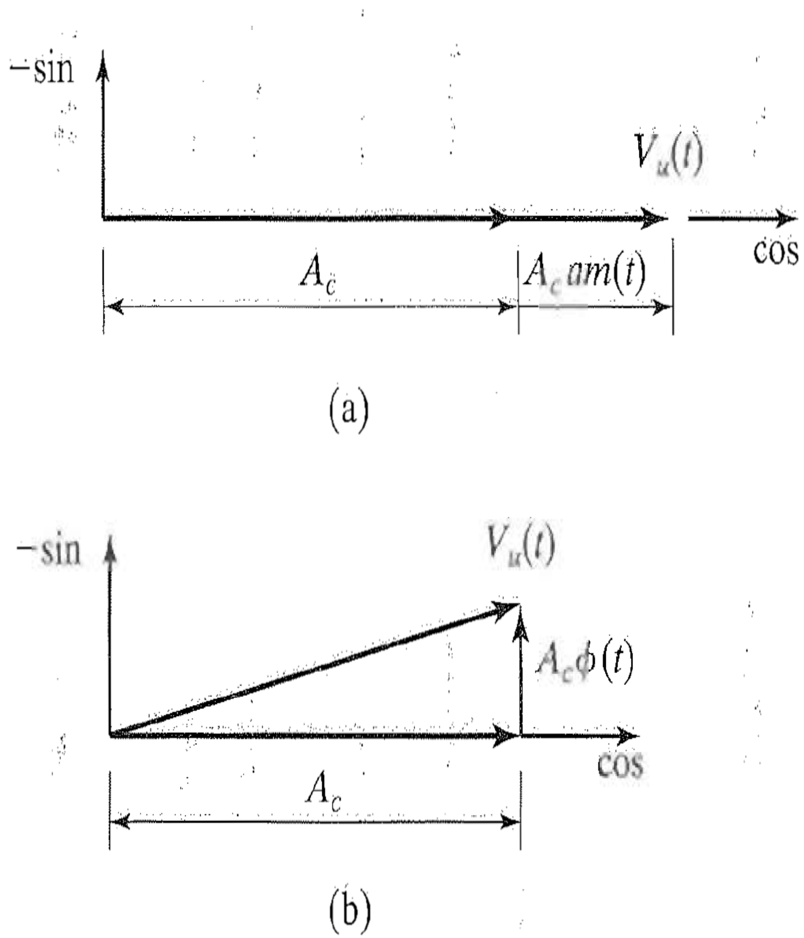


Figure 4.3 Phasor diagram for the conventional AM and narrowband angle modulation.

Angle Modulation by a Sinusoidal Signal (1/11)

- Consider $u(t)=A_c\cos(2\pi f_c t+\beta\sin 2\pi f_m t)$, where β is the modulation index that can by either β_p or β_f
- Using Euler's relation, the modulated signal can be written as

$$u(t)=\mathrm{Re}(A_c e^{j2\pi f_c t} e^{j\beta\sin 2\pi f_m t})$$

Since $e^{j\beta\sin 2\pi f_m t}$ is periodic, its Fourier-series coefficients are obtained from the integral

$$c_n=\frac{1}{2\pi}\int_0^{2\pi} e^{j(\beta\sin u-nu)}du$$

- c_n is a well-known integral called the *Bessel function of the first kind of order n* and is denoted as $J_n(\beta)$

Angle Modulation by a Sinusoidal Signal (2/11)

- We have the Fourier series for the complex exponential as

$$e^{j\beta\sin(2\pi f_m t)} = \sum_{n=-\infty}^{\infty} J_n(\beta) e^{j2\pi n f_m t} \qquad (4.2.4)$$

- By substituting (4.2.4) into $u(t)$, we obtain

$$u(t) = \mathrm{Re}\left(A_c \sum_{n=-\infty}^{\infty} J_n(\beta) e^{j2\pi n f_m t} e^{j2\pi f_c t}\right)$$

$$= \sum_{n=-\infty}^{\infty} A_c J_n(\beta) \cos(2\pi(f_c + n f_m)t)$$

- Even in this very simple case where the modulating signal is a sinusoid of frequency f_m, the angle-modulated signal contains all frequencies of the form $f_c + n f_m$ for $n = 0, \pm 1, \pm 2, \ldots$
- However, the amplitude of the sinusoidal components of frequencies $f_c \pm n f_m$ for large n is very small

Angle Modulation by a Sinusoidal Signal (3/11)

- For small β, we have

$$J_n(\beta) \cong \frac{\beta^n}{2^n n!}$$

We can also verify the following symmetry properties of the Bessel function:

$$J_{-n}(\beta) = \begin{cases} J_n(\beta), & n \text{ even} \\ -J_n(\beta), & n \text{ odd} \end{cases}$$

Angle Modulation by a Sinusoidal Signal (4/11)

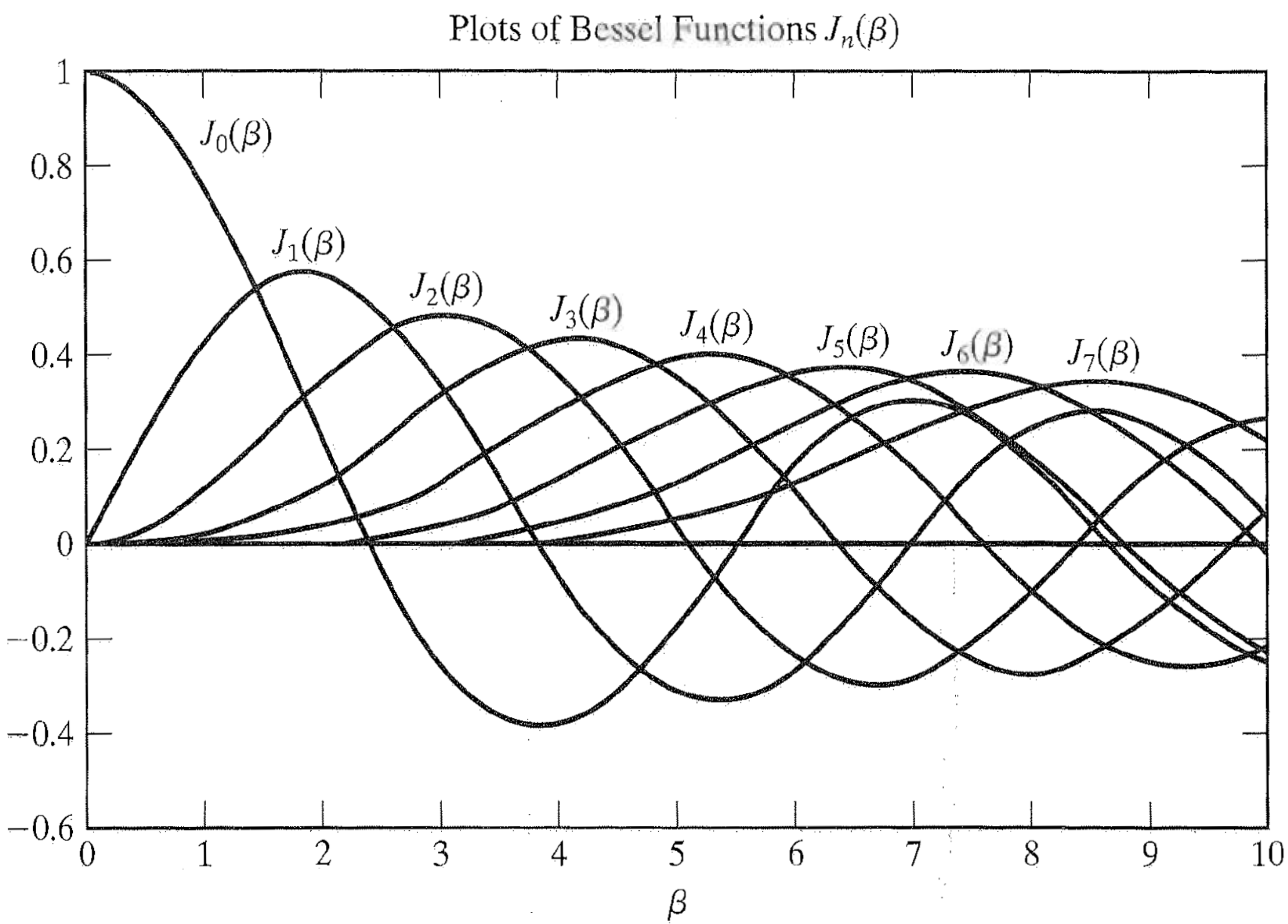


Figure 4.4 Bessel functions for various values of n.

Angle Modulation by a Sinusoidal Signal (5/11)

TABLE 4.1 TABLE OF BESSEL FUNCTION VALUES

n	$\beta = 0.1$	$\beta = 0.2$	$\beta = 0.5$	$\beta = 1$	$\beta = 2$	$\beta = 5$	$\beta = 8$	$\beta = 10$
0	0.997	0.990	0.938	0.765	0.224	−0.178	0.172	−0.246
1	0.050	0.100	0.242	0.440	0.577	−0.328	0.235	0.043
2	0.001	0.005	0.031	0.115	0.353	0.047	−0.113	0.255
3				0.020	0.129	0.365	−0.291	0.058
4				0.002	0.034	0.391	−0.105	−0.220
5					0.007	0.261	0.186	−0.234
6					0.001	0.131	0.338	−0.014
7						0.053	0.321	0.217
8						0.018	0.223	0.318
9						0.006	0.126	0.292
10						0.001	0.061	0.207
11							0.026	0.123
12							0.010	0.063
13							0.003	0.029
14							0.001	0.012
15								0.004
16								0.001

Single and double underlines indicate the number of harmonics containing 70% and 98% of total power, respectively.

Angle Modulation by a Sinusoidal Signal (6/11)

- **Example 4.2.1.** Let the carrier be given by $c(t)=10\cos(2\pi f_c t)$, and let the message signal be $\cos(20\pi t)$. Further assume that the message is used to frequency modulate the carrier with $k_f=50$. Find the expression for the modulated signal and determine how many harmonics should be selected to contain 99% of the modulated-signal power.
- The power content of the carrier signal is given by

$$P_c = \frac{A_c^2}{2} = 50.$$

The modulated signal is represented by

$$u(t) = 10\cos\left(2\pi f_c t + 2\pi k_f \int_{-\infty}^{t} \cos(20\pi\tau)\,d\tau\right)$$

$$= 10\cos\left(2\pi f_c t + 5\sin(20\pi t)\right)$$

Angle Modulation by a Sinusoidal Signal (7/11)

- **Example 4.2.1. (Cont'd)** The modulation index is given by

$$\beta_f = \frac{k_f \max[|m(t)|]}{W} = 5.$$

Thus, the FM-modulated signal is

$$u(t) = \sum_{n=-\infty}^{\infty} 10 J_n(5) \cos(2\pi(f_c + 10n)t).$$

- The frequency content of the modulated signal is concentrated at frequencies of the form $f_c + 10n$ for various n. To make sure that at least 99% of the total power is within the effective bandwidth, we must choose a k large enough such that

$$\sum_{n=-k}^{k} \frac{100 J_n^2(5)}{2} \geq 0.99 \times 50$$

Angle Modulation by a Sinusoidal Signal (8/11)

- By trial-and-error, we find that the smallest value of k is k=6
- This means that if the modulated signal is passed through an ideal bandpass filter centered at f_c with a bandwidth of at least 120 Hz, only 1% of the signal power will be eliminated

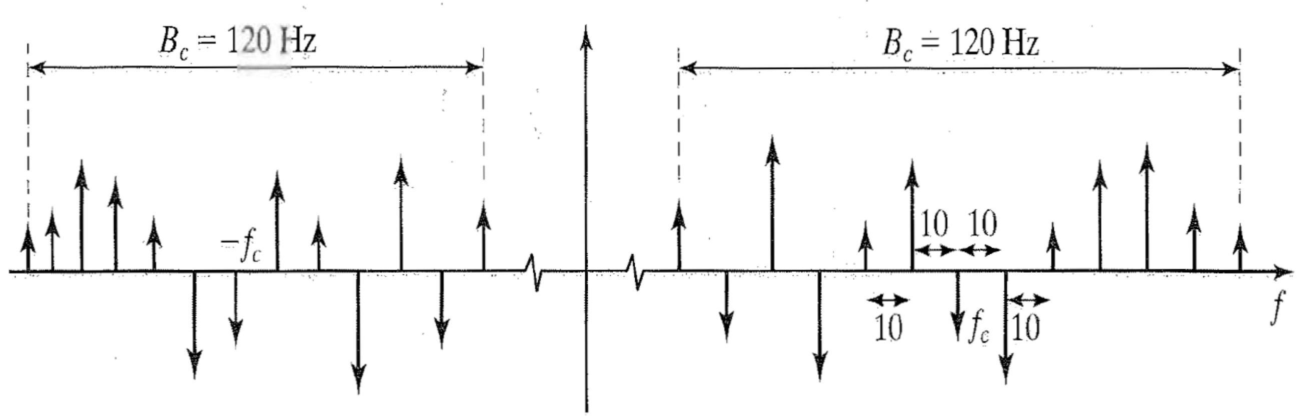


Figure 4.5 The harmonics present inside the effective bandwidth of Example 4.2.1.

Angle Modulation by a Sinusoidal Signal (9/11)

- The effective bandwidth of an angle-modulated signal, which contains at least 98% of the signal power, is given by the relation

$$B_c = 2(\beta + 1) f_m \qquad (4.2.14)$$

- β is the modulation index and f_m is the frequency of the sinusoidal message signal
- Let the message signal is given by $m(t) = a\cos(2\pi f_m t)$. Using (4.2.14), the bandwidth of the modulated signal is given by

$$B_c = 2(\beta + 1) f_m = \begin{cases} 2(k_p a + 1) f_m, & PM \\ 2(\frac{k_f a}{f_m} + 1) f_m, & FM \end{cases}$$

or

$$B_c = \begin{cases} 2(k_p a + 1) f_m, & PM \\ 2(k_f a + f_m), & FM \end{cases}$$

Angle Modulation by a Sinusoidal Signal (10/11)

- Increasing a, the amplitude of the modulating signal, in PM and FM has almost the same effect on increasing the bandwidth B_c
- Increasing f_m, the frequency of the message signal, has a more profound effect in increasing the bandwidth of a PM signal as compared to an FM signal

Angle Modulation by a Sinusoidal Signal (11/11)

- If the number of harmonics in B_c is denoted by M_c, we have

$$M_c = 2(\lfloor \beta + 1 \rfloor) + 1 = 2(\lfloor \beta \rfloor + 1) + 1 = \begin{cases} 2\lfloor k_p a \rfloor + 3, & PM \\ 2\left\lfloor \frac{k_f a}{f_m} \right\rfloor + 3, & FM \end{cases}$$

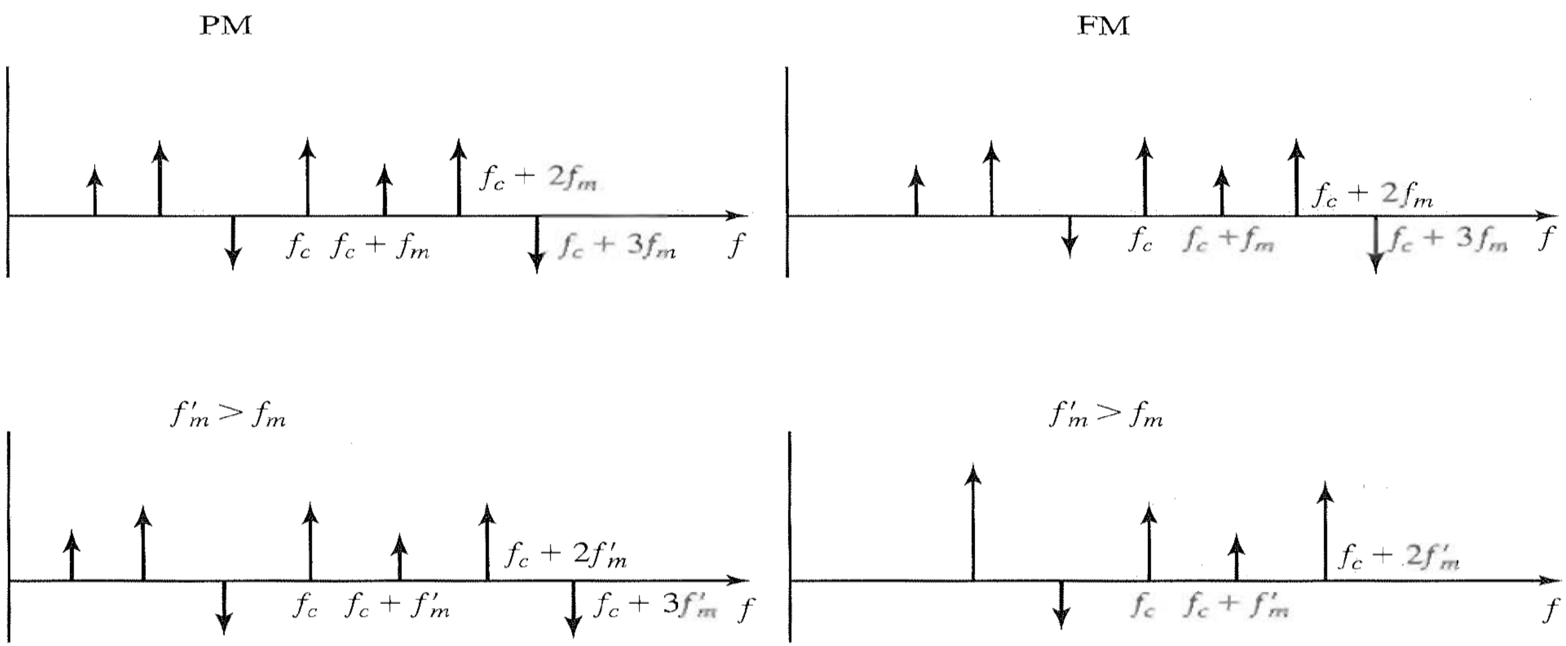


Figure 4.6 The effect of doubling the bandwidth of the message in FM and PM.

Angle Modulation by an Arbitrary Message Signal (1/3)

- The spectral characteristics of an angle-modulated signal for a general message signal $m(t)$ is quite involved due to the nonlinear nature of the modulation process
- The effective bandwidth of an angle-modulated signal can be well approximated as *Carson's rule* and is given by

$$B_c = 2(\beta + 1)W$$

where β is the modulation index defined as

$$\beta = \begin{cases} k_p \max[|\, m(t)\, |], & PM \\ \frac{k_f \max[|m(t)|]}{W}, & FM \end{cases},$$

and W is the bandwidth of the message signal $m(t)$

Angle Modulation by an Arbitrary Message Signal (2/3)

- Since wideband FM has a β with a value that is usually around 5 or more, the bandwidth of an angle-modulated signal is much greater than the bandwidth of various amplitude-modulated schemes
- The bandwidth is either W (in SSB) or $2W$ (in DSB or conventional AM)
- **Example 4.2.2.** Assuming that $m(t)=10\text{sinc}(10^4 t)$, determine the transmission bandwidth of an FM-modulated signal with $k_f=4000$

Angle Modulation by an Arbitrary Message Signal (3/3)

- **Example 4.2.2. (Cont'd)**

For FM, we have $B_c = 2(\beta+1)W$. To find, we have to find the spectrum of $m(t)$. We have $M(f) = 10^{-3}\Pi(10^{-4}f)$, which shows that $m(t)$ has a bandwidth of 5000 Hz. We have

$$\beta = \frac{k_f \max[|m(t)|]}{W} = \frac{4000 \times 10}{5000} = 8$$

and

$$B_c = 2(8+1) \times 5000 = 90 \ kHz$$

Implementation of Angle Modulators and Demodulators (1/1)

- Consider a modulator system with the message signal $m(t)$ as the input and with the modulated signal $u(t)$ as the output; this system has frequencies in its output that were not present in the input
- A modulator (and demodulator) cannot be modeled as a linear time-invariant system, because a linear time-invariant system cannot produce any frequency components in the output that are not present in the input signal
- Angle modulators are generally time-varying and nonlinear systems

Angle Modulators (1/5)

- One method for directly generating an FM signal is to design an oscillator whose frequency changes with the input voltage. When the input voltage is zero, the oscillator generates a sinusoid with frequency f_c; when the input voltage changes, this frequency changes accordingly
- There are two approaches to designing such an oscillator. One is called a voltage-controlled oscillator (VCO) and another one is to use a *varactor diode*
- A varactor diode is a capacitor whose capacitance changes with the applied voltage

Angle Modulators (2/5)

- L_0 is the inductance of the inductor in the tuned circuit
- The capacitance of the varactor diode is given by

$$C_v(t) = C_0 + k_0 m(t)$$

- When $m(t)=0$, the frequency of the tuned circuit is given by

$$f_c = \frac{1}{2\pi\sqrt{L_0 C_0}}$$

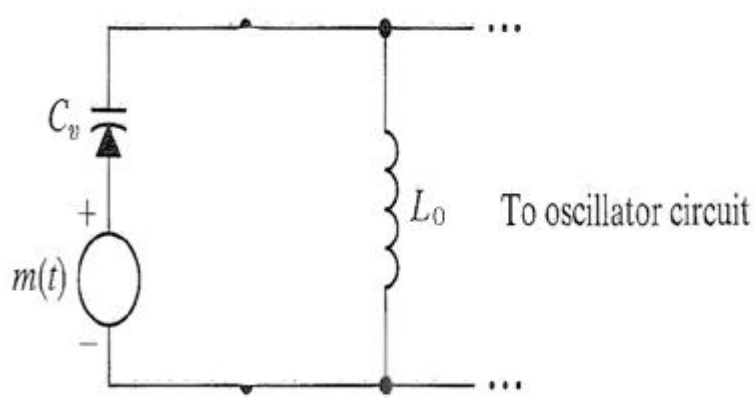


Figure 4.7 Varactor-diode implementation of an angle modulator.

Angle Modulators (3/5)

- For nonzero $m(t)$, we have the oscillation frequency

$$f_i(t) = \frac{1}{2\pi\sqrt{L_0 C_v}}$$
$$= \frac{1}{2\pi\sqrt{L_0(C_0 + k_0 m(t))}}$$
$$= \frac{1}{2\pi\sqrt{L_0 C_0}} \frac{1}{\sqrt{1 + \frac{k_0}{C_0} m(t)}}$$
$$= f_c \frac{1}{\sqrt{1 + \frac{k_0}{C_0} m(t)}}$$

- Assuming that $\varepsilon = \frac{k_0}{C_0} m(t) << 1$, we obtain

$$f_i(t) \approx f_c\left(1 - \frac{k_0}{2C_0} m(t)\right)$$

which is the relation for a frequency-modulated signal

Angle Modulators (4/5)

- Another approach for generating an angle-modulated signal is to generate a narrowband angle-modulated signal and then change it to a wideband signal

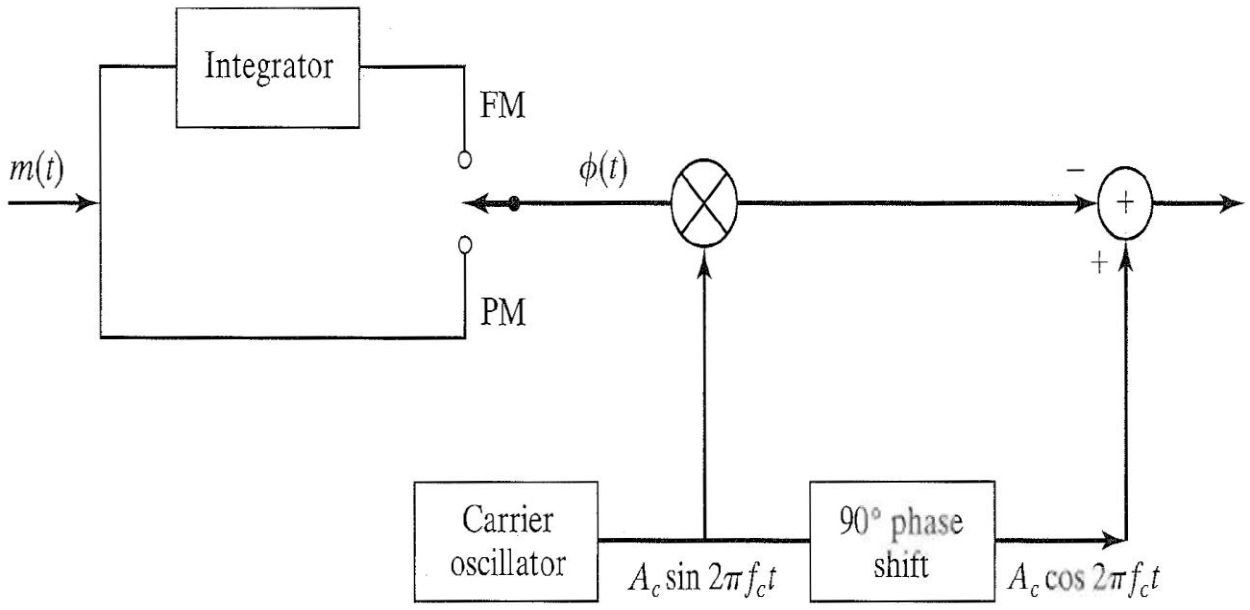


Figure 4.8 Generation of a narrowband angle-modulated signal.

Angle Modulators (5/5)

- A frequency multiplier is usually done by applying the input signal to a nonlinear element and then passing its output through a bandpass filter tuned to the desired center frequency

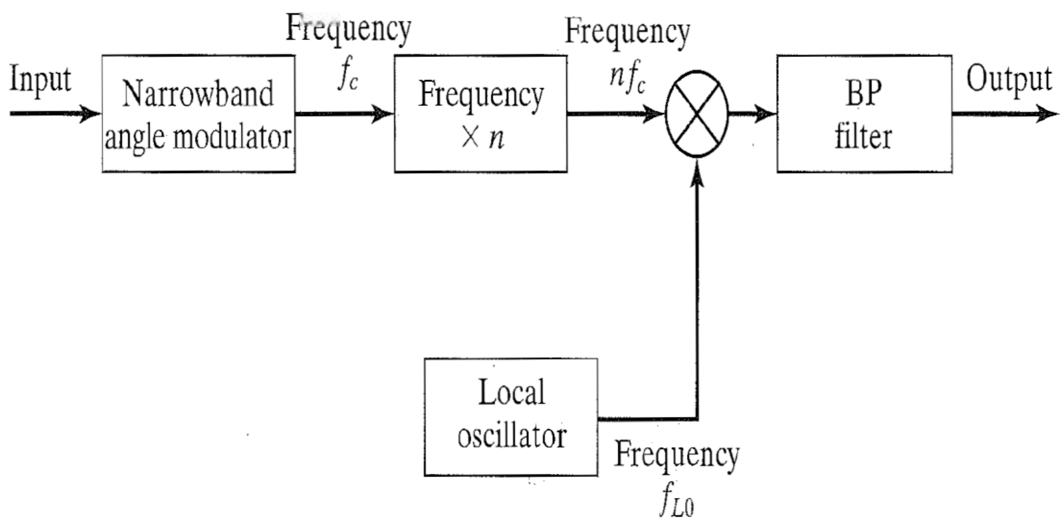


Figure 4.9 Indirect generation of angle-modulated signals.

Angle Demodulators (1/13)

- FM demodulators are implemented by generating an AM signal, whose amplitude is proportional to the instantaneous frequency of the FM signal, and then using an AM demodulator to recover the message signal
- If the frequency response of an LTI system is given by

$$|H(f)| = V_0 + k(f - f_c) \quad for \quad |f - f_c| < \tfrac{B_c}{2}$$

and if the input to the system is

$$u(t) = A_c \cos\left(2\pi f_c t + 2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau\right)$$

then the output will be the signal

$$v_o(t) = A_c(V_o + kk_f m(t))\cos\left(2\pi f_c t + 2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau\right).$$

The next step is to obtain $A_c(V_o + kk_f m(t))$

Angle Demodulators (2/13)

- A general FM demodulator is shown below

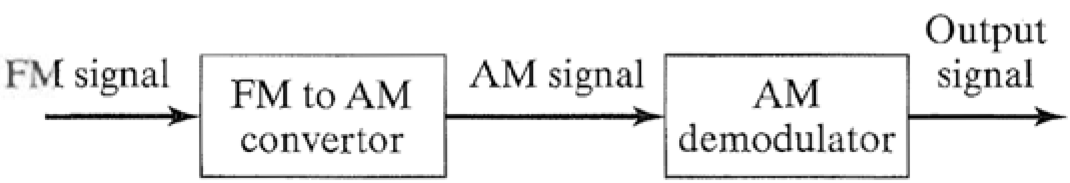


Figure 4.10 A general FM demodulator.

Angle Demodulators (3/13)

- Many circuits can be used to implement the first stage of an FM demodulator, *i.e.*, FM to AM conversion. One such candidate is a simple differentiator with

$$|H(f)| = 2\pi f$$

- Another candidate is the rising half of the frequency characteristics of a tuned circuit, as shown in Fig. 4.11

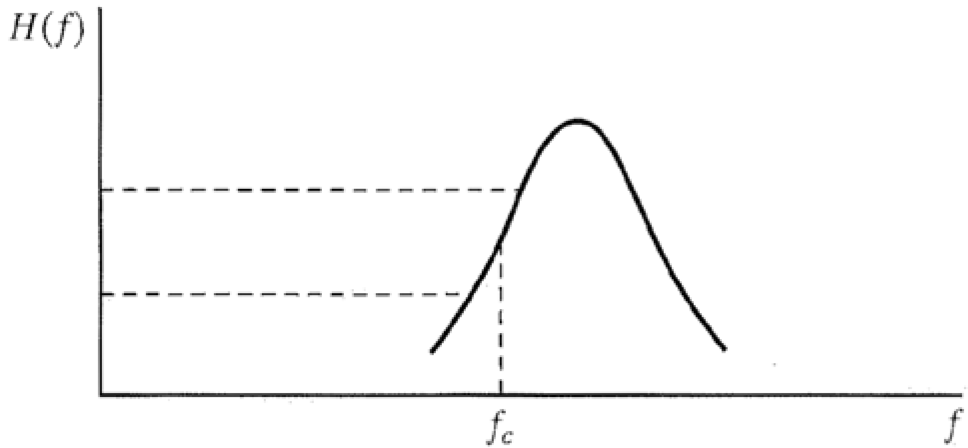
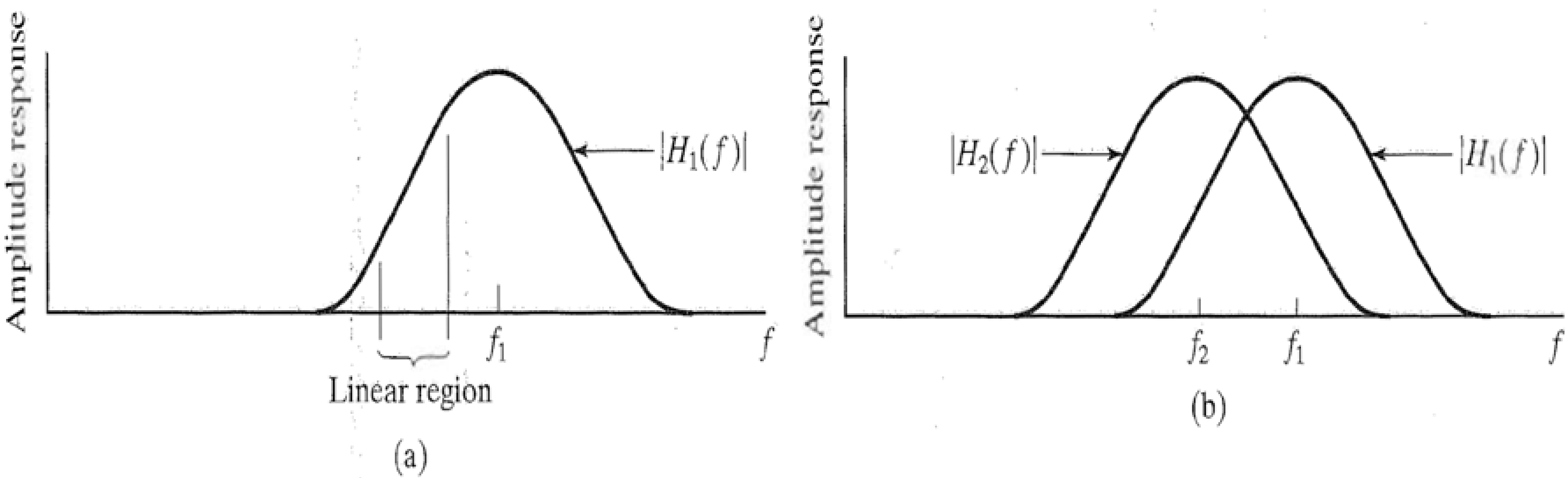


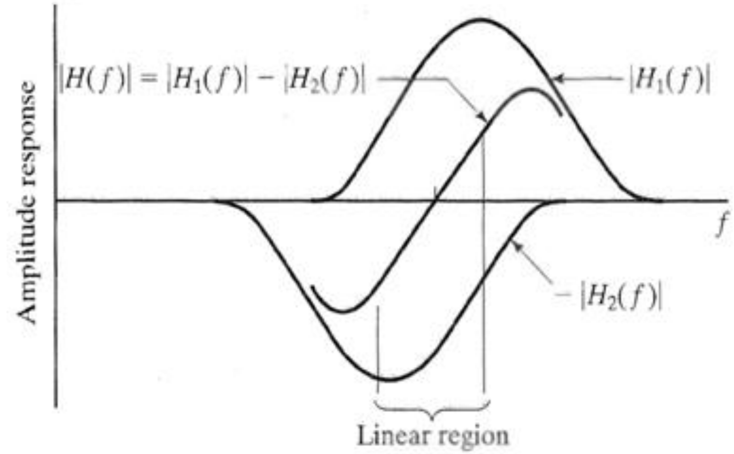
Figure 4.11 A tuned circuit used in an FM demodulator.

Angle Demodulators (4/13)

- To obtain linear characteristics over a wide range of frequencies, usually two circuits tuned at two frequencies f_1 and f_2 are connected in a configuration, which is known as a *balanced discriminator*



Angle Demodulators (5/13)



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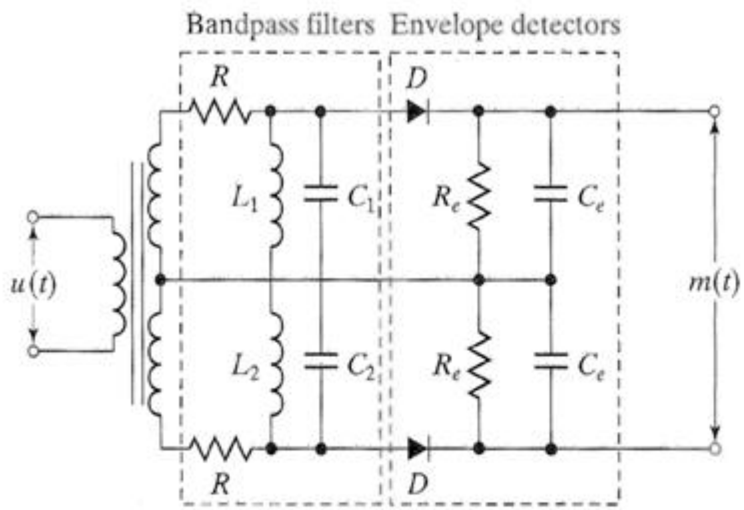

(d)

Figure 4.12 A balanced discriminator and the corresponding frequency response.

Angle Demodulators (6/13)

- These FM-demodulation methods, which transform the FM signal into an AM signal, have a bandwidth equal to the channel bandwidth B_c occupied by the FM signal. Consequently, the noise that is passed by the demodulator is the noise contained within B_c
- An alternative to FM demodulation is the use of a phase-locked loop (PLL)

Angle Demodulators (7/13)

- The input to the PLL is the angle-modulated signal

$$u(t) = A_c \cos[2\pi f_c t + \phi(t)],$$

where, for FM,

$$\phi(t) = 2\pi k_f \int_{-\infty}^{t} m(\tau) d\tau$$

- The VCO generates a sinusoid of a fixed frequency; in this case, it generates the carrier frequency f_c, in the absence of an input control voltage

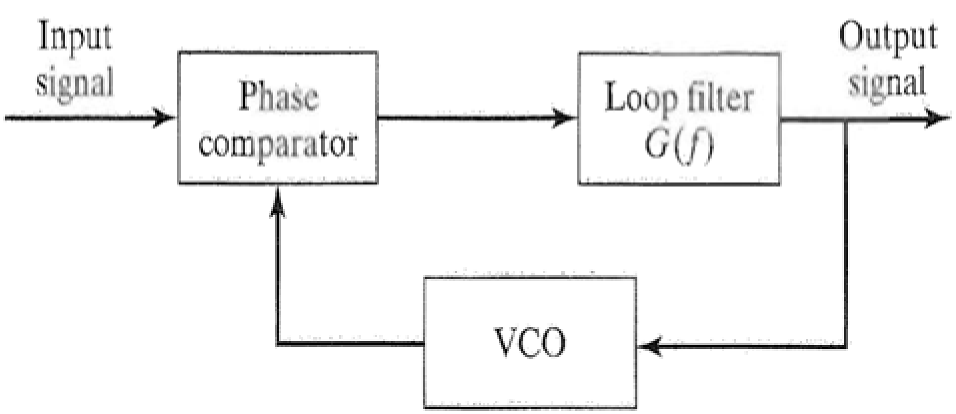

Figure 4.14 Block diagram of PLL-FM demodulator.

Angle Demodulators (8/13)

- Suppose the control voltage to the VCO is the loop filter's output, denoted as $v(t)$. Then, the instantaneous frequency of the VCO is

$$f_v(t) = f_c + k_v v(t),$$

where k_v is a deviation constant with units of Hz/volt

- The VCO output may be expressed as

$$y_v(t) = A_v \sin[2\pi f_c t + \phi_v(t)],$$

where

$$\phi_v(t) = 2\pi k_v \int_0^t v(\tau) d\tau$$

Angle Demodulators (9/13)

- The phase comparator is basically a multiplier and a filter that rejects the signal component centered at $2f_c$. Hence, its output may be expressed as

$$e(t) = \tfrac{1}{2} A_v A_c \sin[\phi(t) - \phi_v(t)],$$

  where the difference $\phi(t) - \phi_v(t) \equiv \phi_e(t)$ constitutes the phase error
- The signal $e(t)$ is the input to the loop filter
- A linearized PLL is present in Fig. 4.15

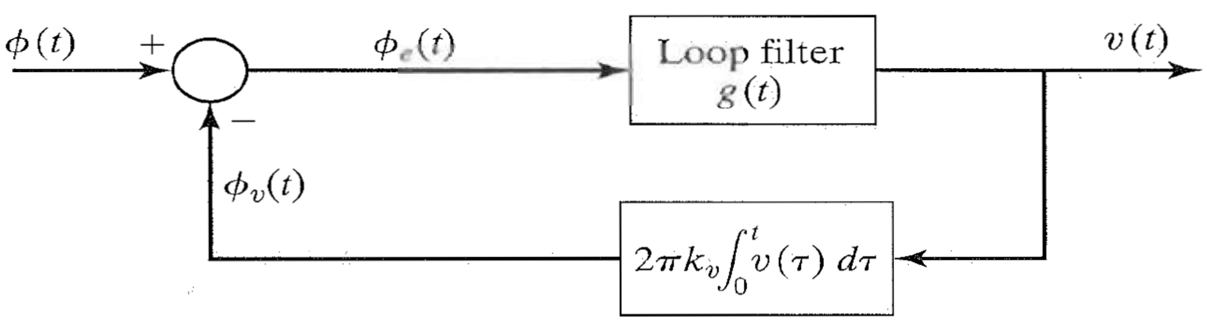

Figure 4.15 Linearized PLL.

Angle Demodulators (10/13)

- Let the PLL be in lock position, so the phase error is small. Then,

$$\sin[\phi(t) - \phi_v(t)] \approx \phi(t) - \phi_v(t) = \phi_e(t)$$

- We may express the phase error as

$$\phi_e(t) = \phi(t) - 2\pi k_v \int_0^t v(\tau) d\tau,$$

or equivalently, either as

$$\frac{d}{dt}\phi_e(t) + 2\pi k_v v(t) = \frac{d}{dt}\phi(t),$$

or as

$$\frac{d}{dt}\phi_e(t) + 2\pi k_v \int_0^\infty \phi_e(\tau) g(t-\tau) d\tau = \frac{d}{dt}\phi(t). \quad (4.3.22)$$

- $g(t)$ denotes the impulse response of the loop filter

Angle Demodulators (11/13)

- The Fourier transform of the integro-differential equation in Eq. (4.3.22) is

$$(j2\pi f)\Phi_e(f) + 2\pi k_v \Phi_e(f) G(f) = (j2\pi f)\Phi(f);$$

hence,

$$\Phi_e(f) = \frac{1}{1+\left(\frac{k_v}{jf}\right)G(f)}\Phi(f).$$

- The corresponding equation for the control voltage to the VCO is

$$V(f) = \Phi_e(f)G(f)$$

$$= \frac{G(f)}{1+\left(\frac{k_v}{jf}\right)G(f)}\Phi(f). \qquad (4.3.25)$$

Angle Demodulators (12/13)

- Suppose that we design *G(f)* such that

$$|k_v \frac{G(f)}{jf}| >> 1$$

in the frequency band $|f| < W$ of the message signal. From (4.3.25), we have

$$V(f) = \frac{G(f)}{\left(\frac{k_v}{jf}\right)G(f)}\Phi(f)$$
$$= \frac{1}{\left(\frac{k_v}{jf}\right)}\Phi(f)$$
$$= \frac{jf}{k_v}\Phi(f),$$

or equivalently, $v(t) = \frac{1}{2\pi k_v}\frac{d}{dt}\phi(t) = \frac{k_f}{k_v}m(t).$

Angle Demodulators (13/13)

- We observe that the output of the loop filter is the desired message signal
- The bandwidth of $G(f)$ should be the same as the bandwidth W of the message signal. Hence, the noise at the output of the loop filter is also limited to the bandwidth W
- The output from the VCO is a wideband FM signal with an instantaneous frequency that follows the instantaneous frequency of the received FM signal