# Chapter 2 Signals and Linear Systems (V)

# Filter Design (1/7)

- Filters are widely used to separate desired signals from undesired signals and interference
- The desired filter characteristics are specified in the frequency domain, in terms of the desired magnitude and phase response of the filter
- In filter design, we determine the coefficients of a causal filter that closely approximates the desired frequency response specifications
- There are a variety of filter types, both analog and digital. We are particularly interested in the design of digital filters, because they are easily implemented in software on a computer

# Filter Design (2/7)

- Digital filters are generally classified as having either a finite duration impulse response (FIR) or an infinite duration impulse response (IIR)
- An FIR filter characterized in the z-domain can be represented by the system function

$$H(z) = \sum_{k=0}^{M-1} h(k) z^{-k},$$

  where $\{h(k), 0 \leqq k \leqq M\text{-}1\}$ is the impulse response of the filter
- The frequency response of the filter is obtained by evaluating $H(z)$ on the unit circle, i.e., by substituting $z=e^{j\omega}$ in $H(z)$ to yield $H(\omega)$

# Filter Design (3/7)

- In the discrete-time domain, the FIR filter is characterized by the (difference) equation

$$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k)$$

where $\{x(n)\}$ is the input sequence to the filter and $y(n)$ is the output sequence

- An IIR filter has both poles and zeros, and it is generally characterized in the $z$-domain by the rational system function

$$H(z) = \frac{\sum_{k=0}^{M-1} b(k)z^{-k}}{1-\sum_{k=1}^{N} a(k)z^{-k}}$$

$\{a(k)\}$ and $\{b(k)\}$ are the filter coefficients

# Filter Design (4/7)

- The frequency response $H(\omega)$ is obtained by evaluating $H(z)$ on the unit circle
- In the discrete-time domain, the IIR filter is characterized by the difference equation

$$y(n) = \sum_{k=1}^{N} a(k) y(n-k) + \sum_{k=0}^{M-1} b(k) x(n-k)$$

# Filter Design (5/7)

- In practice, FIR filters are employed in filtering problems where there is a requirement of a linear-phase characteristic within the pass band of the filter
- If there is no requirement for a linear-phase characteristic, either an IIR or an FIR filter may be employed
- In general, if some phase distortion is either tolerable or unimportant, an IIR filter is preferable, primarily because the implementation involves fewer coefficients and consequently has a lower computational complexity

# Filter Design (6/7)

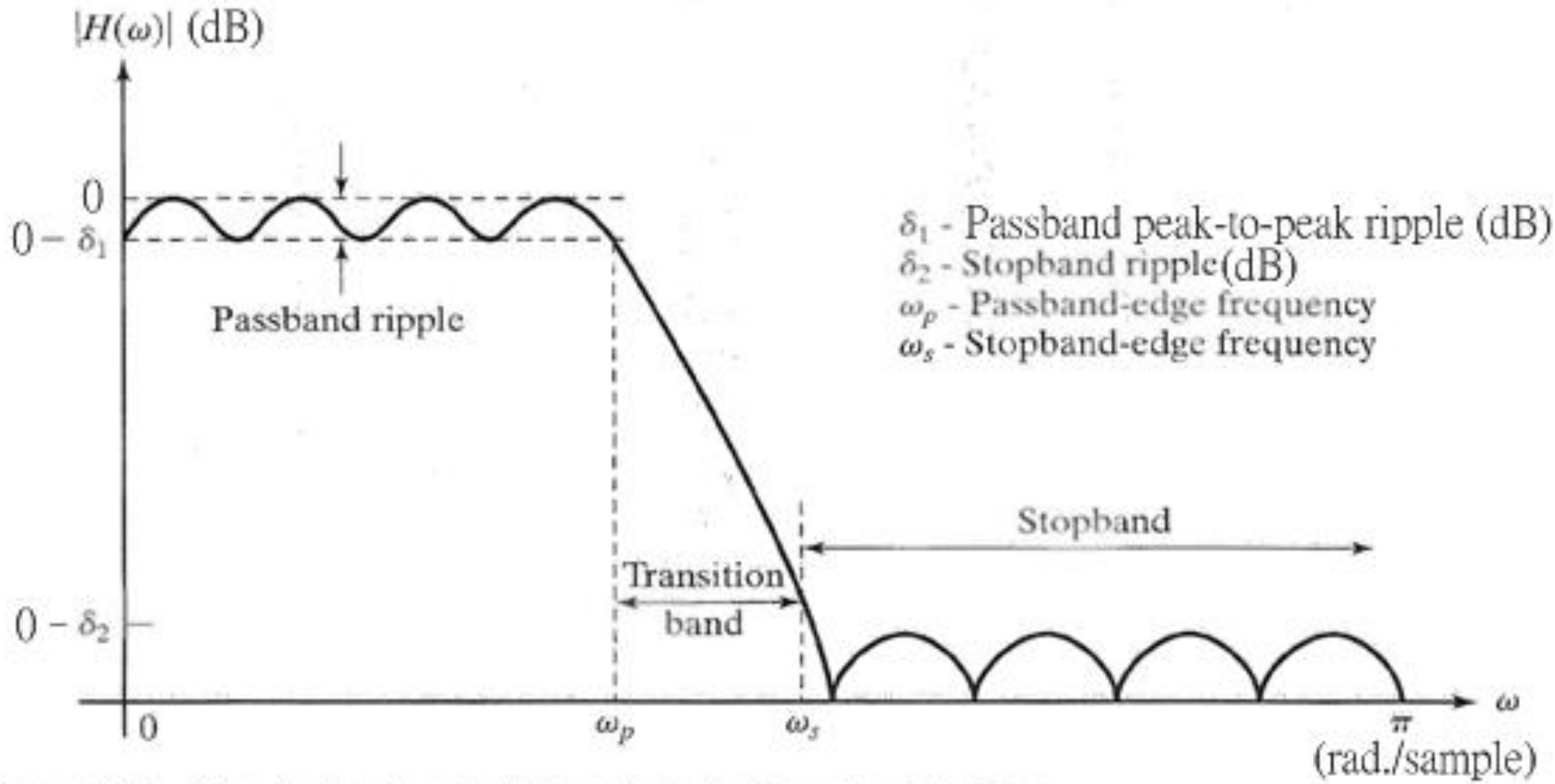


**Figure 2.48** Magnitude characteristics of physically realizable filters.

# Filter Design (7/7)

- In a filter design problem, we usually specify several filter parameter
  - The maximum tolerable passband ripple
  - The maximum tolerable stopband ripple
  - The passband-edge frequency $f_p$
  - The stopband-edge frequency $f_s$
- On the basis of these specifications, we select the filter coefficients that are closest to the desired frequency response specifications

# Power and Energy (1/2)

- The energy and power of a signal represent the energy or power delivered by the signal when it is interpreted as a voltage or current source feeding a 1-ohm resistor
- The energy content of a (generally complex-valued) signal $x(t)$ is defined as

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

and the power content of a signal is

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 \, dt$$

- A signal is energy-type if $E_x<\infty$, and it is power-type if $0<P_x<\infty$

# Power and Energy (2/2)

- A signal cannot be both power- and energy-type because $P_x$=0 for energy-type signals and $E_x$=∞ for power-type signals
- A signal can be neither energy-type nor power-type
- Practically, all periodic signals are power-type and have power

$$P_x = \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} |x(t)|^2 \, dt$$

where $T_0$ is the period and $\alpha$ is any arbitrary real number

# Energy-Type Signals (1/8)

- For an energy-type signal $x(t)$, we define the autocorrelation function

$$R_x(\tau) = x(\tau) \star x^*(-\tau)$$
$$= \int_{-\infty}^{\infty} x(t)x^*(t-\tau)dt$$
$$= \int_{-\infty}^{\infty} x(t+\tau)x^*(t)dt$$

- By setting $\tau=0$, we obtain its energy content, i.e.,

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$
$$= R_x(0)$$

- Using the autocorrelation property of the Fourier transform (see Sec. 2.3.2), we derive the Fourier transform of $R_x(\tau)$ to be $|X(f)|^2$. (Hint: $\mathscr{F}[x^*(-\tau)] = X^*(f)$)

# Energy-Type Signals (2/8)

- Employing Rayleigh's theorem, we have

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$
$$= \int_{-\infty}^{\infty} |X(f)|^2 \, df$$

- If we pass the signal $x(t)$ through a filter with the (generally complex) impulse response $h(t)$ and frequency response $H(f)$, the output will be $y(t) = x(t) \bigstar h(t)$ and in the frequency domain $Y(f) = X(f)H(f)$
- The energy content of the output signal $y(t)$ is

$$E_y = \int_{-\infty}^{\infty} |y(t)|^2 \, dt$$
$$= \int_{-\infty}^{\infty} |Y(f)|^2 \, df$$
$$= R_y(0)$$

# Energy-Type Signals (3/8)

- The inverse Fourier transform of $|Y(f)|^2$ is

$$
\begin{aligned}
R_y(\tau) &= \mathscr{F}^{-1}[|Y(f)|^2] \\
&= \mathscr{F}^{-1}[|X(f)|^2 |H(f)|^2] \\
&= \mathscr{F}^{-1}[|X(f)|^2] \bigstar \mathscr{F}^{-1}[|H(f)|^2] \\
&= R_x(\tau) \bigstar R_h(\tau)
\end{aligned}
$$

- Now let us assume that

$$
H(f) = \begin{cases} 1 & W < f < W + \Delta W \\ 0 & \text{otherwise} \end{cases}
$$

Then

$$
|Y(f)|^2 = \begin{cases} |X(f)|^2 & W < f < W + \Delta W \\ 0 & \text{otherwise} \end{cases}
$$

# Energy-Type Signals (4/8)

- We thus have

$$E_y = \int_{-\infty}^{\infty} |Y(f)|^2 \, df$$
$$\approx |X(W)|^2 \, \Delta W$$

This means that $|X(W)|^2 \Delta W$ is the amount of energy in $x(t)$, which is located in the bandwidth $[W, W+\Delta W]$.

- This shows why $|X(f)|^2$ is called the *energy spectral density* of a signal $x(t)$, and why it represents the amount of energy per unit bandwidth present in the signal at various frequencies
- We define the energy spectral density (or energy spectrum of the signal $x(t)$) as

$$\mathscr{G}_x(f) = |X(f)|^2 = \mathscr{F}[R_x(\tau)]$$

# Energy-Type Signals (5/8)

- To summarize,
  - For any energy-type signal $x(t)$, we define the autocorrelation function $R_x(\tau) = x(\tau) \star x^*(-\tau)$
  - The energy spectral density of $x(t)$, denoted by $\mathcal{G}_x(f)$, is the Fourier transform of $R_x(\tau)$. It is equal to $|X(f)|^2$
  - The energy content of $x(t)$, $E_x$, is the value of the autocorrelation function evaluated at $\tau=0$ or, equivalently, the integral of the energy spectral density over all frequencies, i.e.,

$$\begin{aligned} E_x &= R_x(0) \\ &= \int_{-\infty}^{\infty} \mathcal{G}_x(f)\, df \end{aligned}$$

# Energy-Type Signals (6/8)

- To summarize, (Cont'd)
  - If $x(t)$ is passed through a filter with the impulse response $h(t)$ and the output is denoted by $y(t)$, we have

$$y(t) = x(t) \star h(t)$$

$$R_y(\tau) = R_x(\tau) \star R_h(\tau)$$

$$\mathcal{G}_y(f) = \mathcal{G}_x(f)\,\mathcal{G}_h(f) = |X(f)|^2 |H(f)|^2$$

# Energy-Type Signals (7/8)

- **Example 2.5.1.** Determine the autocorrelation function, energy spectral density, and energy content of the signal $x(t)=e^{-\alpha t}u_{-1}(t)$, $\alpha>0$
- First we find the Fourier transform of $x(t)$

$$X(f)=\frac{1}{\alpha+j2\pi f}$$

Hence,

$$\mathcal{G}_x(f)=|X(f)|^2=\frac{1}{\alpha^2+(2\pi f)^2}$$

and

$$R_x(\tau)=\mathcal{F}^{-1}[|X(f)|^2]=\frac{1}{2\alpha}e^{-\alpha|\tau|}$$

The energy content is

$$E_x=\frac{1}{2\alpha}$$

# Energy-Type Signals (8/8)

- **Example 2.5.2.** If the signal in the preceding example is passed through a filter with impulse response $h(t)=e^{-\beta t}u_{-1}(t)$, $\beta>0$, $\beta\neq\alpha$, determine the autocorrelation function, the energy spectral density, and the energy content of the signal at the output
- The frequency response of the filter is

$$H(f)=\frac{1}{\beta+j2\pi f}.$$

Therefore,

$$|Y(f)|^2=|X(f)|^2|H(f)|^2.$$

Note that

$$R_y(\tau)=\mathscr{F}^{-1}[|Y(f)|^2]$$

and

$$E_y=R_y(0)$$

# Power Type Signals (1/7)

- We define *time-average autocorrelation function* of the power-type signal $x(t)$ as

$$R_x(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)x^*(t-\tau)dt$$

- The power content of the signal can be obtained from

$$\begin{aligned} P_x &= \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2\, dt \\ &= R_x(0) \end{aligned}$$

- We define the *power-spectral density* or the *power spectrum* of $x(t)$ to be the Fourier transform of $R_x(\tau)$

$$S_x(f) = \mathscr{F}[R_x(\tau)]$$

# Power Type Signals (2/7)

- We can express the power content of the signal $x(t)$ in terms of $S_x(f)$ by noting that

$$P_x = R_x(0)$$
$$= \int_{-\infty}^{\infty} S_x(f)\,df$$

- If a power-type signal is passed through a filter with impulse response $h(t)$, the output is

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

and the time-average autocorrelation function for the output signal is

$$R_y(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} y(t)\,y^*(t-\tau)\,dt$$

# Power Type Signals (3/7)

- Substituting for $y(t)$, we obtain

$$R_y(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \left[ \int_{-\infty}^{\infty} h(u)x(t-u)du \right] \left[ \int_{-\infty}^{\infty} h^*(v)x^*(t-\tau-v)dv \right] dt$$

By making a change of variable $w=t-u$ and changing the order of integration, we obtain

$$R_y(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(u)h^*(v)$$

$$\times \lim_{T\to\infty} \int_{-\frac{T}{2}-u}^{\frac{T}{2}+u} \left[ x(w)x^*(u+w-\tau-v)dw \right] dudv$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_x(\tau+v-u)h(u)h^*(v)dudv$$

$$= \int_{-\infty}^{\infty} [R_x(\tau+v) \bigstar h(\tau)]h^*(v)dv = \int_{-\infty}^{\infty} R_x(\tau+v)h^*(v)dv \bigstar h(\tau)$$

$$= R_x(\tau) \bigstar h^*(-\tau) \bigstar h(\tau) \quad \text{(Please think why?)}$$

# Power Type Signals (4/7)

- Taking the Fourier transform of both sides of the above equation, we obtain

$$\begin{aligned} S_y(f) &= S_x(f) H(f) H^*(f) \\ &= S_x(f) \,|\, H(f) \,|^2 \end{aligned}$$

- $S_x(f)$ represents the amount of power at various frequencies
- Assume the signal $x(t)$ is a periodic signal with the period $T_0$ and has the Fourier-series coefficients $\{x_n\}$. We have

$$\begin{aligned} R_x(\tau) &= \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) x^*(t-\tau) dt \\ &= \lim_{k\to\infty} \frac{1}{kT_0} \int_{-\frac{kT_0}{2}}^{\frac{kT_0}{2}} x(t) x^*(t-\tau) dt \\ &= \lim_{k\to\infty} \frac{k}{kT_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t) x^*(t-\tau) dt = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t) x^*(t-\tau) dt \end{aligned}$$

# Power Type Signals (5/7)

- If we substitute the Fourier-series expansion of the periodic signal in this relation, we obtain

$$R_x(\tau) = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x_n x_m{}^* e^{j2\pi\frac{m}{T_0}\tau} e^{j2\pi\frac{n-m}{T_0}t} dt$$

Now, using the fact that

$$\frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} e^{j2\pi\frac{n-m}{T_0}t} dt = \delta_{mn}$$

we obtain

$$R_x(\tau) = \sum_{n=-\infty}^{\infty} |x_n|^2 e^{j2\pi\frac{n}{T_0}\tau}$$

- We see that the time-average autocorrelation function of a periodic signal is itself periodic

# Power Type Signals (6/7)

- To determine the power-spectral density of a periodic signal, we can simply find the Fourier series of $x(t)$
- We expect the power is concentrated at discrete frequencies (the harmonics). The power spectral density of a periodic signal is given by

$$S_x(f) = \sum_{n=-\infty}^{\infty} |x_n|^2 \, \delta(f - \frac{n}{T_0})$$

- To find the power content of a periodic signal, we must integrate over the whole frequency spectrum; we obtain

$$P_x = \sum_{n=-\infty}^{\infty} |x_n|^2$$

# Power Type Signals (7/7)

- If this periodic signal passes through an LTI system with the frequency response *H*(*f*), the output will be periodic and the power spectral density of the output can be obtained by employing the relation between the power spectral densities of the input and the output of a filter. Thus,

$$S_y(f) = |H(f)|^2 \sum_{n=-\infty}^{\infty} |x_n|^2 \, \delta(f - \frac{n}{T_0})$$

$$= \sum_{n=-\infty}^{\infty} |x_n|^2 |H(\frac{n}{T_0})|^2 \, \delta(f - \frac{n}{T_0})$$

- The power content of the output signal is

$$P_y = \sum_{n=-\infty}^{\infty} |x_n|^2 \, |H(\frac{n}{T_0})|^2$$

# Hilbert Transform and its Properties (1/5)

- The Hilbert transform of a signal $x(t)$ is a signal $\hat{x}(t)$ whose frequency components lag the frequency components of $x(t)$ by $90^{o}$
- The Hilbert transform is unlike many other transforms because it does not involve a change of a domain
- The Hilbert transform is not equivalent to the original signal, rather it is a completely different signal
- The Hilbert transform does not involve a domain change, i.e., the Hilbert transform of a signal $x(t)$ is another signal denoted by $\hat{x}(t)$ in the same domain

# Hilbert Transform and its Properties (2/5)

- A delay of $\pi/2$ at all frequencies means that $e^{j2\pi f_0 t}$ will become $e^{j(2\pi f_0 t-\pi/2)}=-je^{j2\pi f_0 t}$ and $e^{-j2\pi f_0 t}$ will become $e^{-j(2\pi f_0 t-\pi/2)}=je^{-j2\pi f_0 t}$. In other words, at positive frequencies, the spectrum of the signal is multiplied by $-j$; at negative frequencies, it is multiplied by $+j$. This is equivalent to saying that the spectrum (Fourier transform) of the signal is multiplied by $-j\mathrm{sgn}(f)$.
- We assume $x(t)$ is real and has no DC component, *i.e.*, $X(f)\,|_{f=0}=0$
- We have

$$\mathscr{F}[\hat{x}(t)]=-j\mathrm{sgn}(f)X(f)$$

# Hilbert Transform and its Properties (3/5)

- The Hilbert transform of $x(t)=A\cos(2\pi f_0 t+\theta)$ is
  $\hat{x}(t) = A\cos(2\pi f_0 t+\theta-90^{o}) = A\sin(2\pi f_0 \mathrm{t}+\theta)$

- Using Table 2.1, we have

$$\mathscr{F}^{-1}[-j\mathrm{sgn}(f)]=1/\pi t$$

Hence,

$$\hat{x}(t)=\frac{1}{\pi t}\bigstar x(t)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x(\tau)}{t-\tau}d\tau.$$

Thus, the operation of the Hilbert transform is equivalent to a convolution, *i.e.*, filtering

# Hilbert Transform and its Properties (4/5)

- **Example. 2.6.1.** Determine the Hilbert transform of the signal $x(t)=2\text{sinc}(2t)$
- We use the frequency-domain approach to solve the problem. We have,

$$\mathscr{F}[x(t)] = \Pi(\tfrac{f}{2}) = \Pi(f+\tfrac{1}{2}) + \Pi(f-\tfrac{1}{2}).$$

The first term contains all the negative frequencies and the second term contains all the positive frequencies

- We use the relation $\mathscr{F}[\hat{x}(t)] = -j\,\text{sgn}(f)\mathscr{F}[x(t)]$, which results in

$$\mathscr{F}[\hat{x}(t)] = j\Pi(f+\tfrac{1}{2}) - j\Pi(f-\tfrac{1}{2})$$

Taking the inverse Fourier transform, we have

$$\hat{x}(t) = 2\sin(\pi t)\,\text{sin}\,c(t)$$

# Hilbert Transform and its Properties (5/5)

- $x(t)$ and $y(t)$ are orthogonal if

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = 0$$

# Hilbert Transform and its Properties – Evenness and Oddness (1/1)

- The Hilbert transform of an even and real signal is odd. The Hilbert transform of an odd and real signal is even.
- If $x(t)$ is even and real signal, then $X(f)$ is real and even function; therefore, $-j\operatorname{sgn}(f)X(f)$ is an imaginary and odd function. Hence, its inverse Fourier transform $\hat{x}(t)$ is odd and real
- If $x(t)$ is odd and real, then $X(f)$ is imaginary and odd; thus $-j\operatorname{sgn}(f)X(f)$ is real and even. Therefore, $\hat{x}(t)$ is even and real

# Hilbert Transform and its Properties – Sign Reversal (1/1)

- Applying the Hilbert-transform operation to a signal twice causes a sign reversal of the signal, i.e.,

$$\hat{\hat{x}}(t) = -x(t)$$

- Since

$$\mathcal{F}[\hat{x}(t)] = -j\operatorname{sgn}(f)X(f)$$

$$\mathcal{F}[\hat{\hat{x}}(t)] = -j\operatorname{sgn}(f)\mathcal{F}[\hat{x}(t)] = [-j\operatorname{sgn}(f)]^2 X(f)$$

thus,

$$\mathcal{F}[\hat{\hat{x}}(t)] = -X(f),$$

$$\hat{\hat{x}}(t) = -x(t)$$

# Hilbert Transform and its Properties – Energy (1/1)

- The energy content of a real signal is equal to the energy content of its Hilbert transform
- Using Rayleigh's theorem of the Fourier transform, we have

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |X(f)|^2 \, df$$

and

$$E_{\hat{x}} = \int_{-\infty}^{\infty} |\hat{x}(t)|^2 \, dt = \int_{-\infty}^{\infty} |-j\,\mathrm{sgn}(f) X(f)|^2 \, df.$$

Using the fact that $|-j\mathrm{sgn}(f)|^2 = 1$ except for $f = 0$, and the fact that $X(f)$ does not contain any impulses at the origin completes the proof

# Hilbert Transform and its Properties – Orthogonality (1/1)

- The real signal $x(t)$ and its Hilbert transform are orthogonal
- Using Parseval's theorem of the Fourier transform, we obtain

$$
\begin{aligned}
\int_{-\infty}^{\infty} x(t)(\hat{x}(t))^* dt &= \int_{-\infty}^{\infty} X(f)[-j\,\mathrm{sgn}(f)X(f)]^* df \\
&= \int_{-\infty}^{\infty} X(f)[-j\,\mathrm{sgn}(f)]^* X^*(f) df \\
&= \int_{-\infty}^{\infty} [-j\,\mathrm{sgn}(f)]^* |X(f)|^2 df \\
&= \int_{-\infty}^{0} (-j)|X(f)|^2 df + \int_{0}^{\infty} (j)|X(f)|^2 df \\
&= 0
\end{aligned}
$$

Note, in the last step, we have used the fact that $X(f)$ is Hermitian; therefore, $|X(f)|^2$ is even

# Lowpass and Bandpass Signals (1/7)

- A lowpass signal is a signal in which the spectrum (frequency content) of the signal is located around the zero frequency
- A bandpass signal is a signal with a spectrum far from the zero frequency. The frequency spectrum of a bandpass signal is usually located around a frequency $f_c$, which is much higher than the bandwidth of the signal
- The bandwidth of the bandpass signal is usually much less than the frequency $f_c$, which is close to the location of the frequency content

# Lowpass and Bandpass Signals (2/7)

- The extreme case of a bandpass signal is a single frequency signal whose frequency is equal to $f_c$. The bandwidth of this signal is zero. It can be written as

$$x(t) = A\cos(2\pi f_c t + \theta)$$

- This sinusoidal signal can be represented by a phasor

$$x_l = Ae^{j\theta}$$

$A$ is positive and the range of $\theta$ is $-\pi$ to $\pi$

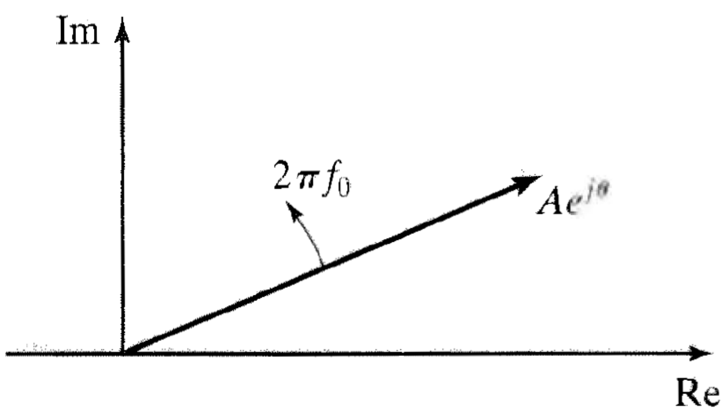


**Figure 2.50** Phasor corresponding to a sinusoid signal.

# Lowpass and Bandpass Signals (3/7)

- The projection of the phasor on the real axis is

$$x(t)=A\cos(2\pi f_c t+\theta)$$

- We can expand the signal $x(t)$ as

$$\begin{aligned} x(t) &= A\cos(2\pi f_c t+\theta) \\ &= A\cos(\theta)\cos(2\pi f_c t) - A\sin(\theta)\sin(2\pi f_c t) \\ &= x_c\cos(2\pi f_c t) - x_s\sin(2\pi f_c t) \end{aligned}$$

  $x_c=A\cos(\theta)$ is called the *in-phase component*. The other component $x_s=A\sin(\theta)$ is called *quadrature component*

- We can also write

$$x_l=Ae^{j\theta}=x_c+jx_s$$

# Lowpass and Bandpass Signals (4/7)

- Assume that we have a phasor with slowly varying magnitude and phase. This is represented by

$$x_l(t) = A(t)e^{j\theta(t)},$$

where $A(t)$ and $\theta(t)$ are slowly varying signals (compared to $f_c$)

We have

$$\begin{aligned} x(t) &= \mathrm{Re}[A(t)e^{j(2\pi f_c t + \theta(t))}] \\ &= A(t)\cos(\theta(t))\cos(2\pi f_c t) - A(t)\sin(\theta(t))\sin(2\pi f_c t) \\ &= x_c(t)\cos(2\pi f_c t) - x_s(t)\sin(2\pi f_c t) \end{aligned}$$

- This signal contains a range of frequencies; therefore, its bandwidth is not zero

# Lowpass and Bandpass Signals (5/7)

- The spectra of three bandpass signals are shown in Fig. 2.51

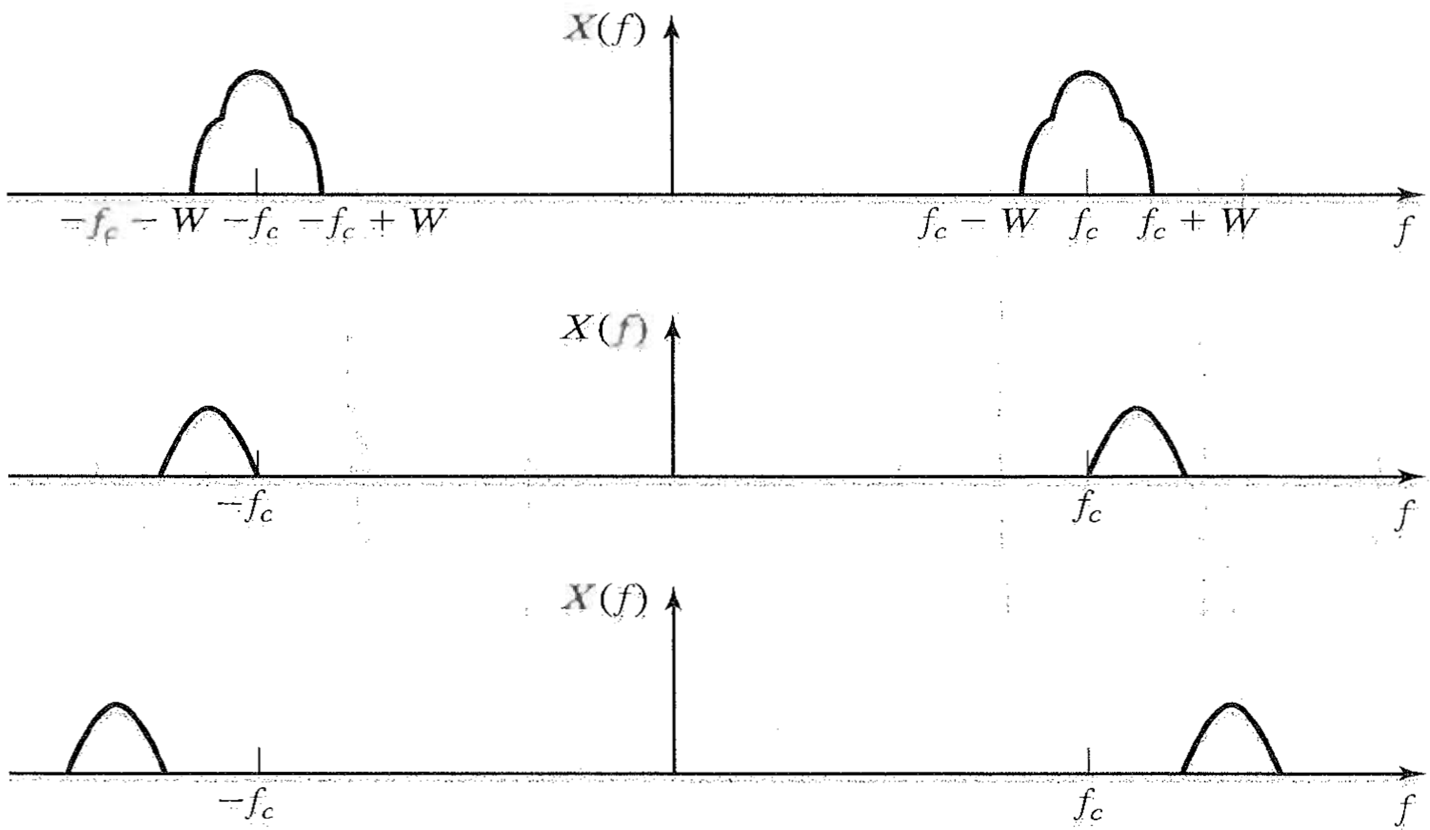


**Figure 2.51** Spectra of three bandpass signals.

# Lowpass and Bandpass Signals (6/7)

- The in-phase and quadrature components are

$$x_c(t) = A(t)\cos(\theta(t))$$

$$x_s(t) = A(t)\sin(\theta(t))$$

and we have

$$x(t) = x_c(t)\cos(2\pi f_c t) - x_s(t)\sin(2\pi f_c t)$$

- Note that both the in-phase and quadrature components of a bandpass signal are slowly varying signals; therefore, they are both lowpass signals
- The complex lowpass signal $x_l(t) = x_c(t) + jx_s(t)$ is called the *lowpass equivalent* of the bandpass signal $x(t)$

# Lowpass and Bandpass Signals (7/7)

- If we represent $x_l(t)$ in polar coordinates, we have

$$x_l(t) = \sqrt{x_c^2(t) + x_s^2(t)}\, e^{j \arctan \frac{x_s(t)}{x_c(t)}}$$

Now if we define the *envelope* and the *phase* of the bandpass signal as

$$\begin{cases} |x_l(t)| = A(t) = \sqrt{x_c^2(t) + x_s^2(t)} \\ \angle x_l(t) = \theta(t) = \arctan \frac{x_s(t)}{x_c(t)}, \end{cases}$$

we can express $x_l(t)$ as

$$x_l(t) = A(t) e^{j\theta(t)}$$

- In summary, we have

$$\begin{aligned} x(t) &= \mathrm{Re}[x_l(t) e^{j2\pi f_c t}] \\ &= \mathrm{Re}[A(t) e^{j(2\pi f_c t + \theta(t))}] \\ &= A(t) \cos(2\pi f_c t + \theta(t)) \end{aligned}$$