Chapter 2 Signals and Linear Systems (IV)

Basic Properties of the Fourier Transform (1/38)

- **Linearity.** The Fourier-transform operation is linear. That is, if $x_1(t)$ and $x_2(t)$ are signals with Fourier transforms $X_1(f)$ and $X_2(f)$ respectively, the Fourier transform of $\alpha x_1(t)+\beta x_2(t)$ is $\alpha X_1(f)+\beta X_2(f)$, where α and β are two arbitrary (real or complex) scalars
- **Example 2.3.4.** Determine the Fourier transform of $u_{-1}(t)$. The unit-step signal can be rewritten as

$$u_{-1}(t) = \begin{cases} \frac{1}{2}+\frac{1}{2}\operatorname{sgn}(t), & t \neq 0 \\ 1 & , \ t = 0 \end{cases}.$$

We have

$$\begin{aligned} \mathscr{F}[u_{-1}(t)] &= \mathscr{F}[\tfrac{1}{2}+\tfrac{1}{2}\operatorname{sgn}(t)] \\ &= \tfrac{1}{2}\delta(f)+\frac{1}{j2\pi f} \end{aligned}$$

Basic Properties of the Fourier Transform (2/38)

- **Duality.** If $X(f)=\mathscr{F}[x(t)]$ then

$$x(f)=\mathscr{F}[X(-t)]$$

and

$$x(-f)=\mathscr{F}[X(t)]$$

- To show this property, we begin with the inverse Fourier-transform relation

$$x(t)=\int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df.$$

Then, we introduce the change of variable $u=-f$ to obtain

$$x(t)=\int_{-\infty}^{\infty} X(-u)e^{-j2\pi ut}du.$$

Let $t=f$, we have

$$x(f)=\int_{-\infty}^{\infty} X(-u)e^{-j2\pi uf}du$$

Basic Properties of the Fourier Transform (3/38)

- Finally, substituting t for u, we get

$$x(f) = \int_{-\infty}^{\infty} X(-t)e^{-j2\pi tf}dt$$

or

$$x(f) = \mathscr{F}[X(-t)].$$

- Using the same technique once more, we obtain

$$x(-f) = \mathscr{F}[X(t)]$$

Basic Properties of the Fourier Transform (4/38)

- **Example 2.3.5.** Determine the Fourier transform of sinc(t).

  Note that $\Pi(t)$ is an even signal and, therefore, that $\Pi(f) = \Pi(-f)$. We can use the duality theorem to obtain

$$\mathscr{F}[\text{sinc}(t)] = \Pi(f) = \Pi(-f)$$

- **Example 2.3.6.** Determine the Fourier transform of $1/t$.

  We already have

$$\mathscr{F}[\text{sgn}(t)] = \frac{1}{j\pi f}$$

  to have

$$\mathscr{F}\left[\frac{1}{j\pi t}\right] = \text{sgn}(-f) = -\text{sgn}(f).$$

  By the linearity theorem, we have

$$\mathscr{F}\left[\frac{1}{t}\right] = -j\pi\,\text{sgn}(f).$$

Basic Properties of the Fourier Transform (5/38)

- **Shift in Time Domain.** A shift of t_0 in the time origin causes a phase shift of $-2\pi f t_0$ in the frequency domain. In other words,

$$\mathscr{F}[x(t-t_0)] = e^{-j2\pi f t_0} \mathscr{F}[x(t)].$$

To prove this, we have

$$\mathscr{F}[x(t-t_0)] = \int_{-\infty}^{\infty} x(t-t_0) e^{-j2\pi f t} dt.$$

With a change of variable of $u = t - t_0$, we obtain

$$\mathscr{F}[x(t-t_0)] = \int_{-\infty}^{\infty} x(u) e^{-j2\pi f t_0} e^{-j2\pi f u} dt$$

$$= e^{-j2\pi f t_0} \int_{-\infty}^{\infty} x(u) e^{-j2\pi f u} dt$$

$$= e^{-j2\pi f t_0} \mathscr{F}[x(t)]$$

Basic Properties of the Fourier Transform (6/38)

- **Example 2.3.7.** Determine the Fourier transform of the signal shown in Fig. 2.37.
- We have

$$x(t) = \Pi(t - \tfrac{3}{2}).$$

By applying the shift theorem, we obtain

$$\mathscr{F}[x(t)] = e^{-j2\pi f \times \frac{3}{2}} \operatorname{sinc}(f) = e^{-j3\pi f} \operatorname{sinc}(f).$$

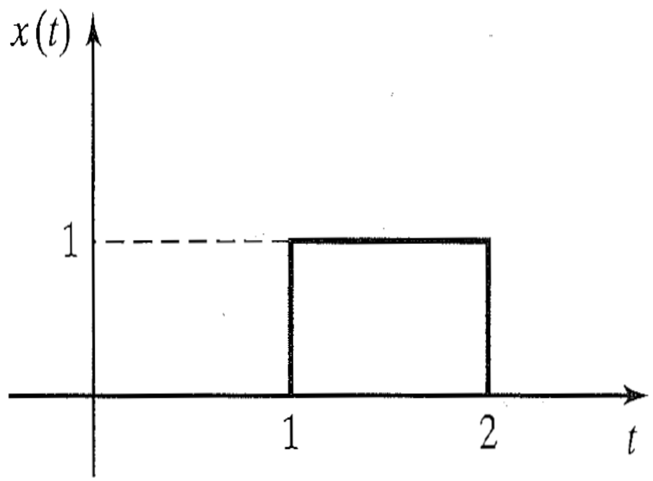

Figure 2.37 Signal $x(t)$.

Basic Properties of the Fourier Transform (7/38)

- **Example 2.3.8.** Determine the Fourier transform of the impulse train

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0).$$

- The Fourier-series expansion of $x(t)$ can be represented as

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} e^{j2\pi \frac{n}{T_0} t}.$$

Taking the Fourier transform of both sides of the above equation, we obtain

$$X(f) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} \delta(f - n\frac{1}{T_0}).$$

If we replace $1/T_0$ with f_0, $X(f)$ can be written as

$$X(f) = f_0 \sum_{n=-\infty}^{\infty} \delta(f - nf_0).$$

Basic Properties of the Fourier Transform (8/38)

- **Scaling.** For any real $a \neq 0$, we have

$$\mathscr{F}[x(at)] = \frac{1}{|a|} X\left(\frac{f}{a}\right).$$

- To see this, we note that

$$\mathscr{F}[x(at)] = \int_{-\infty}^{\infty} x(at) e^{-j2\pi ft} dt$$

and make the change in variable $u = at$. Then,

$$\mathscr{F}[x(at)] = \frac{1}{|a|} \int_{-\infty}^{\infty} x(u) e^{-j2\pi fu/a} du$$

$$= \frac{1}{|a|} X\left(\frac{f}{a}\right)$$

- Note that in the pervious expression, if $|a| > 1$, then $x(at)$ is a contracted form of $x(t)$, whereas if $|a| < 1$, $x(at)$ is an expanded version of $x(t)$

Basic Properties of the Fourier Transform (9/38)

- If we expand a signal in the time domain, its frequency-domain representation (Fourier transform) contracts; if we contract a signal in the time domain, its frequency domain representation expands
- Since contracting a signal in the time domain makes the changes in the signal more abrupt, thus increasing its frequency content

Basic Properties of the Fourier Transform (10/38)

- **Example 2.3.9.** Determine the Fourier transform of the signal

$$x(t) = \begin{cases} 3 & 0 \le t \le 4 \\ 0 & otherwise. \end{cases}$$

- $x(t)$ can be represented as $x(t) = 3\Pi\left(\frac{t-2}{4}\right)$. Using the linearity, time shift, and scaling properties, we have

$$\mathscr{F}[3\Pi\left(\frac{t-2}{4}\right)] = 3e^{-4j\pi f}\mathscr{F}[\Pi\left(\frac{t}{4}\right)]$$

$$= 12e^{-j4\pi f}\,\mathrm{sinc}(4f)$$

Basic Properties of the Fourier Transform (11/38)

- **Convolution.** If the signals $x(t)$ and $y(t)$ both possess Fourier transforms, then

$$\mathscr{F}[x(t)\bigstar y(t)]=\mathscr{F}[x(t)]\bullet\mathscr{F}[y(t)]=X(f)\bullet Y(f)$$

- For a proof, we have

$$\mathscr{F}[x(t)\bigstar y(t)]=\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}x(\tau)y(t-\tau)d\tau\right]e^{-j2\pi ft}dt$$
$$=\int_{-\infty}^{\infty}x(\tau)\left[\int_{-\infty}^{\infty}y(t-\tau)e^{-j2\pi f(t-\tau)}dt\right]e^{-j2\pi f\tau}d\tau.$$

Now with the change of variable $u=t-\tau$, we have

$$\int_{-\infty}^{\infty}y(t-\tau)e^{-j2\pi f(t-\tau)}dt=\int_{-\infty}^{\infty}y(u)e^{-j2\pi fu}du=Y(f)$$

Therefore,

$$\mathscr{F}[x(t)\bigstar y(t)]=\int_{-\infty}^{\infty}x(\tau)[Y(f)]e^{-j2\pi f\tau}d\tau=X(f)\cdot Y(f)$$

Basic Properties of the Fourier Transform (12/38)

- Finding the response of an LTI system to a given input is much easier in the frequency domain than it is the time domain. This theorem is the basis of the frequency-domain analysis of LTI systems
- **Example 2.3.10.** Determine the Fourier transform of the signal $\Lambda(t)$
- It is enough to note that $\Lambda(t) = \Pi(\mathrm{t}) \star \Pi(\mathrm{t})$ and use the convolution theorem.

  We obtain

$$\mathscr{F}[\Lambda(t)] = \mathscr{F}[\Pi(\mathrm{t})] \cdot \mathscr{F}[\Pi(\mathrm{t})] = \mathrm{sinc}^2(f)$$

Basic Properties of the Fourier Transform (13/38)

- **Modulation.** The Fourier transform of $x(t)e^{j2\pi f_0 t}$ is $X(f-f_0)$
- To show this relation, we have

$$\mathscr{F}[x(t)e^{j2\pi f_0 t}] = \int_{-\infty}^{\infty} x(t)e^{j2\pi f_0 t}e^{-j2\pi f t}dt$$

$$= \int_{-\infty}^{\infty} x(t)e^{-j2\pi(f-f_0)t}dt$$

$$= X(f-f_0)$$

- **Example 2.3.12.** Determine the Fourier transform of $x(t)=e^{j2\pi f_0 t}$.

$$\mathscr{F}[e^{j2\pi f_0 t}] = \mathscr{F}[1e^{j2\pi f_0 t}]$$

$$= \delta(f-f_0)$$

Note that since $x(t)$ is not real, its Fourier transform does not have the Hermitian symmetry

Basic Properties of the Fourier Transform (14/38)

- **Example 2.3.13.** Determine the Fourier transform of the signal $\cos(2\pi f_0 t)$

  We have

$$\mathscr{F}[\cos(2\pi f_0 t)] = \mathscr{F}[\tfrac{1}{2} e^{j2\pi f_0 t} + \tfrac{1}{2} e^{-j2\pi f_0 t}]$$
$$= \tfrac{1}{2}\delta(f - f_0) + \tfrac{1}{2}\delta(f + f_0)$$

- **Example 2.3.14.** Determine the Fourier transform of the signal $x(t)\cos(2\pi f_0 t)$

  We have

$$\mathscr{F}[x(t)\cos(2\pi f_0 t)] = \mathscr{F}[\tfrac{1}{2} x(t) e^{j2\pi f_0 t} + \tfrac{1}{2} x(t) e^{-j2\pi f_0 t}]$$
$$= \tfrac{1}{2} X(f - f_0) + \tfrac{1}{2} X(f + f_0)$$

Basic Properties of the Fourier Transform (15/38)

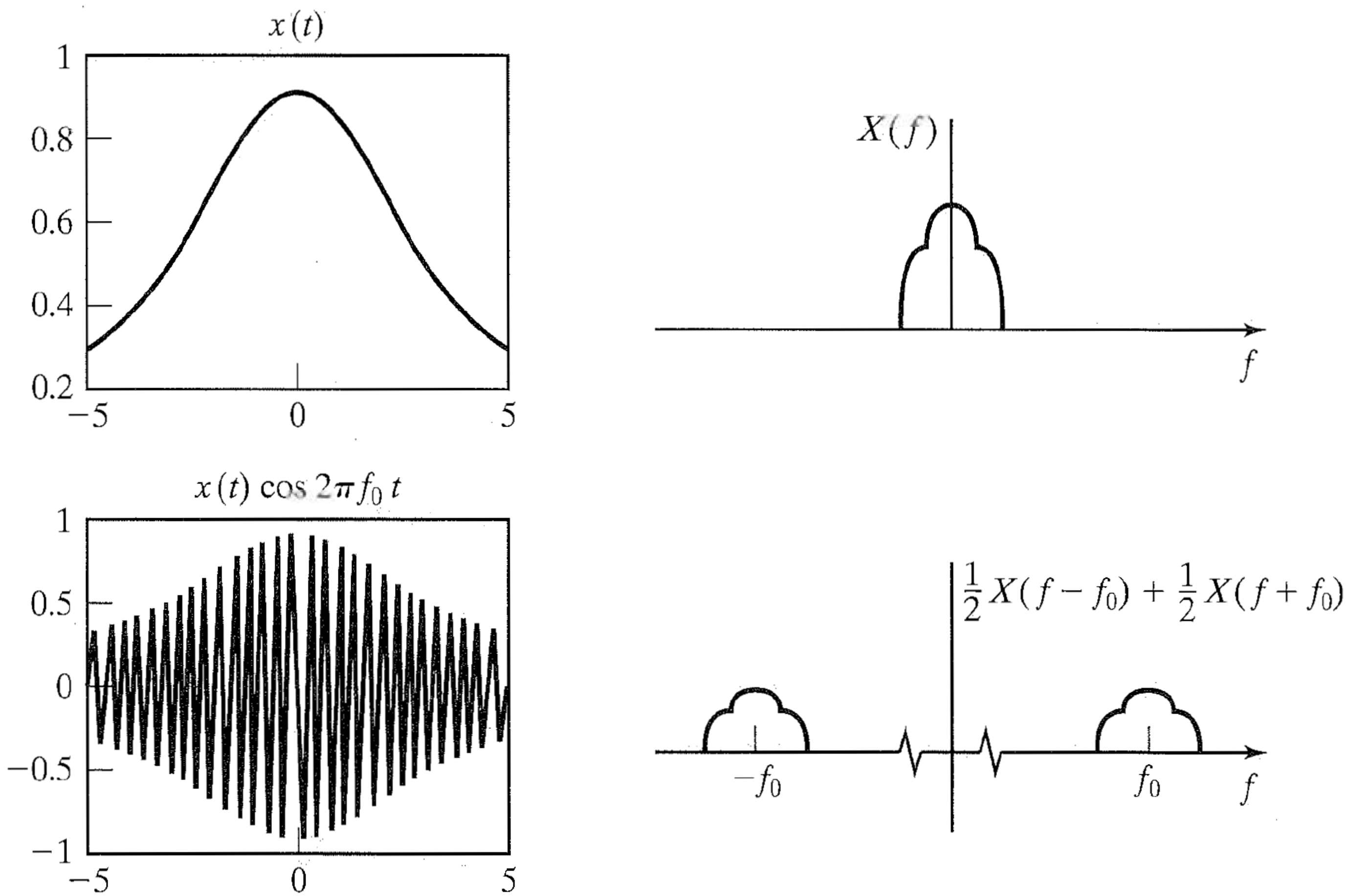

Figure 2.38 Effect of modulation in both the time and frequency domain.

Basic Properties of the Fourier Transform (16/38)

- **Example 2.3.15.** Determine the Fourier transform of the signal

$$x(t) = \begin{cases} \cos(\pi t) & |t| \leq \frac{1}{2} \\ 0 & otherwise \end{cases}$$

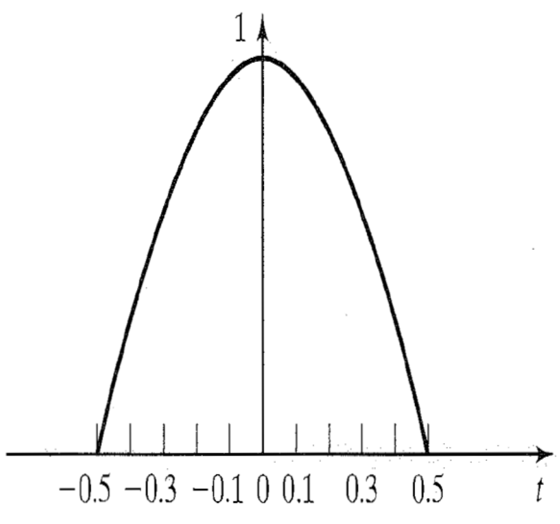

Figure 2.39 Signal $x(t)$.

Basic Properties of the Fourier Transform (17/38)

- **Example 2.3.15. (Cont'd)**

  Note that $x(t)$ can be expressed as

  $$x(t) = \Pi(t)\cos(\pi t).$$

  Therefore,

  $$\mathscr{F}[\Pi(t)\cos(\pi t)] = \tfrac{1}{2}\operatorname{sin}c(f - \tfrac{1}{2}) + \tfrac{1}{2}\operatorname{sin}c(f + \tfrac{1}{2})$$

Basic Properties of the Fourier Transform (18/38)

- **Parseval's Relation.** If the Fourier transforms of the signals $x(t)$ and $y(t)$ are denoted by $X(f)$ and $Y(f)$ respectively, then

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \int_{-\infty}^{\infty} X(f)Y^*(f)df.$$

- **Rayleigh's theorem.** If we substitute $y(t)=x(t)$ into Parseval's relation, we obtain

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-\infty}^{\infty} |X(f)|^2\, df.$$

Basic Properties of the Fourier Transform (19/38)

- **Example 2.3.16.** Use Parseval's relation or Rayleigh's theorem, determine the values of the integrals

$$\int_{-\infty}^{\infty} \operatorname{sinc}^4(t)dt$$

and

$$\int_{-\infty}^{\infty} \operatorname{sinc}^3(t)dt.$$

- We have $\mathscr{F}[\operatorname{sinc}^2(t)]=\Lambda(f)$. Using Rayleigh's theorem with $x(t)=\operatorname{sinc}^2(t)$, we get

$$\begin{aligned}\int_{-\infty}^{\infty} \operatorname{sinc}^4(t)dt &= \int_{-\infty}^{\infty} |\operatorname{sinc}(t)|^4\, dt \\ &= \int_{-\infty}^{\infty} |\Lambda(f)|^2\, df \\ &= \int_{-1}^{0} (f+1)^2 df + \int_{0}^{1} (-f+1)^2 df \\ &= \tfrac{2}{3}\end{aligned}$$

Basic Properties of the Fourier Transform (20/38)

- **Example 2.3.16. (Cont'd)**
- Note that $\mathscr{F}[\text{sinc}(t)] = \Pi(f)$; therefore, by Parseval's theorem, we have

$$\begin{aligned}\int_{-\infty}^{\infty} \text{sinc}^3(t)dt &= \int_{-\infty}^{\infty} \text{sinc}^2(t)\,\text{sinc}(t)dt \\ &= \int_{-\infty}^{\infty} \Pi(f)\Lambda(f)df \\ &= 1\times\tfrac{1}{2} + \tfrac{1}{2}\times\tfrac{1}{2} \\ &= \tfrac{3}{4}\end{aligned}$$

Basic Properties of the Fourier Transform (21/38)

- **Example 2.3.16. (Cont'd)**

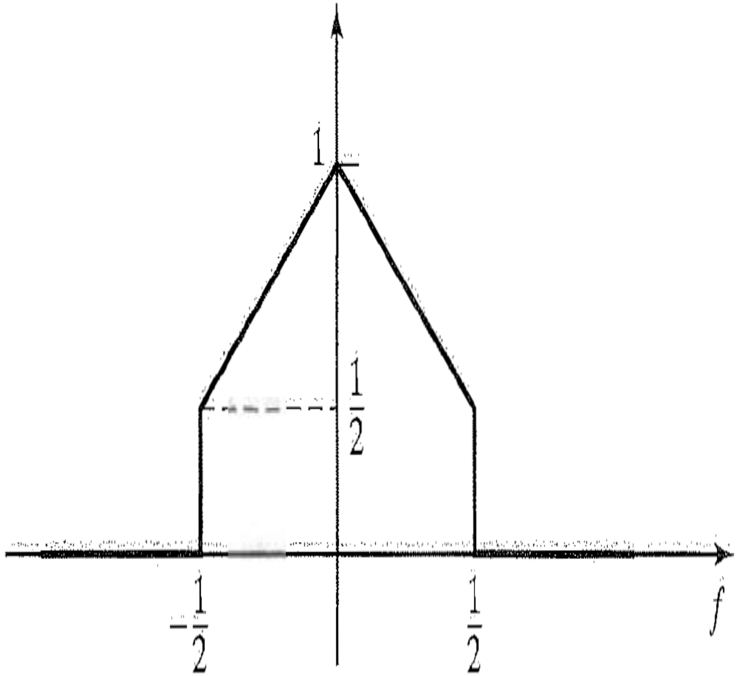

Figure 2.40 Product of Π and Λ.

Basic Properties of the Fourier Transform (22/38)

- **Autocorrelation.** The (time) autocorrelation function of the signal $x(t)$ is denoted by $R_x(\tau)$ and is defined by

$$R_X(\tau) = \int_{-\infty}^{\infty} x(t)x^*(t-\tau)dt.$$

The autocorrelation theorem states that

$$\mathcal{F}[R_x(\tau)] = |X(f)|^2$$

- We note that

$$\begin{aligned} R_X(\tau) &= \int_{-\infty}^{\infty} x(t)x^*(t-\tau)dt \\ &= \int_{-\infty}^{\infty} x(t)x^*(-(\tau-t))dt \\ &= x(\tau) \bigstar x^*(-\tau) \end{aligned}$$

and

$$\mathcal{F}[x^*(-\tau)] = \int_{-\infty}^{\infty} x^*(-\tau)e^{-j2\pi f\tau}d\tau = \int_{-\infty}^{\infty} x^*(u)e^{j2\pi fu}du$$

$$= \left(\int_{-\infty}^{\infty} x(u)e^{-j2\pi fu}du\right)^* = X^*(f)$$

Basic Properties of the Fourier Transform (23/38)

- **Differentiation.** The Fourier transform of the derivative of a signal can be obtained from the relation

$$\mathscr{F}\left[\frac{d}{dt}x(t)\right]=j2\pi fX(f)$$

- To see this, we have

$$\frac{d}{dt}x(t)=\frac{d}{dt}\int_{-\infty}^{\infty}X(f)e^{j2\pi ft}df$$

$$=\int_{-\infty}^{\infty}j2\pi fX(f)e^{j2\pi ft}df.$$

We then conclude that

$$\mathscr{F}^{-1}[j2\pi fX(f)]=\frac{d}{dt}x(t)$$

or

$$\mathscr{F}\left[\frac{d}{dt}x(t)\right]=j2\pi fX(f)$$

Basic Properties of the Fourier Transform (24/38)

- With repeated application of the differentiation theorem, we obtain the relation

$$\mathscr{F}\left[\frac{d^n}{dt^n}x(t)\right]=(j2\pi f)^n X(f)$$

Basic Properties of the Fourier Transform (25/38)

- **Example 2.3.17.** Determine the Fourier transform of the signal shown in Fig. 2.41.

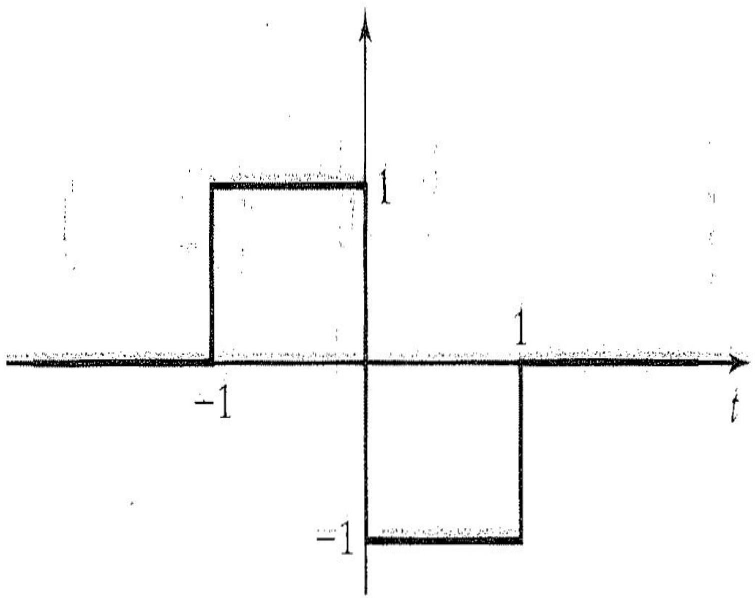

Figure 2.41 Signal $x(t)$.

Basic Properties of the Fourier Transform (26/38)

- **Example 2.3.17. (Cont'd)**
- Obviously, $x(t) = \frac{d}{dt}\Lambda(t)$. Therefore, by applying the differentiation theorem, we have

$$\begin{aligned}\mathscr{F}[x(t)] &= \mathscr{F}\left[\frac{d}{dt}\Lambda(t)\right] \\ &= j2\pi f\,\mathscr{F}[\Lambda(t)] \\ &= j2\pi f\,\mathrm{sinc}^2(f)\end{aligned}$$

Basic Properties of the Fourier Transform (27/38)

- **Differentiation in Frequency Domain.** We begin with

$$\mathscr{F}[tx(t)] = \frac{j}{2\pi}\frac{d}{df}X(f).$$

Repeated use of this theorem yields

$$\mathscr{F}[t^n x(t)] = \left(\frac{j}{2\pi}\right)^n \frac{d^n}{df^n} X(f).$$

- To show this, we have

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$

$$\frac{dX(f)}{df} = \int_{-\infty}^{\infty} x(t) \tfrac{d}{df} e^{-j2\pi ft} dt$$

$$= \int_{-\infty}^{\infty} (-j2\pi t) x(t) e^{-j2\pi ft} dt$$

$$\frac{j}{2\pi}\frac{dX(f)}{df} = \int_{-\infty}^{\infty} t x(t) e^{-j2\pi ft} dt$$

$$\Rightarrow \mathscr{F}[tx(t)] = \frac{j}{2\pi}\frac{d}{df}X(f)$$

Basic Properties of the Fourier Transform (28/38)

- **Example 2.3.18.** Determine the Fourier transform of $x(t)=t$
- Setting $y(t)=1$ and using the relation $\mathscr{F}[ty(t)]=\frac{j}{2\pi}\frac{d}{df}Y(f)$, we have

$$\begin{aligned}\mathscr{F}[ty(t)] &= \mathscr{F}[t] \\ &= \frac{j}{2\pi}\frac{dY(f)}{df} \\ &= \frac{j}{2\pi}\delta'(f)\end{aligned}$$

Basic Properties of the Fourier Transform (29/38)

- **Integration.** The Fourier transform of the integral of a signal can be determined from the relation

$$\mathscr{F}\left[\int_{-\infty}^{t} x(\tau)d\tau\right] = \frac{X(f)}{j2\pi f} + \frac{1}{2}X(0)\delta(f)$$

- To show this, we start with the result of Problem 2.15 to obtain

$$\int_{-\infty}^{t} x(\tau)d\tau = x(t) \bigstar u_{-1}(t).$$

Now using the convolution theorem and the Fourier transform of $u_{-1}(t)$, we have

$$\mathscr{F}\left[\int_{-\infty}^{t} x(\tau)d\tau\right] = X(f)\left[\frac{1}{j2\pi f} + \frac{1}{2}\delta(f)\right]$$

$$= \frac{X(f)}{j2\pi f} + \frac{1}{2}X(0)\delta(f)$$

Basic Properties of the Fourier Transform (30/38)

- **Example 2.3.19.** Determine the Fourier transform of the signal $x(t)$ shown in Fig. 2.42.

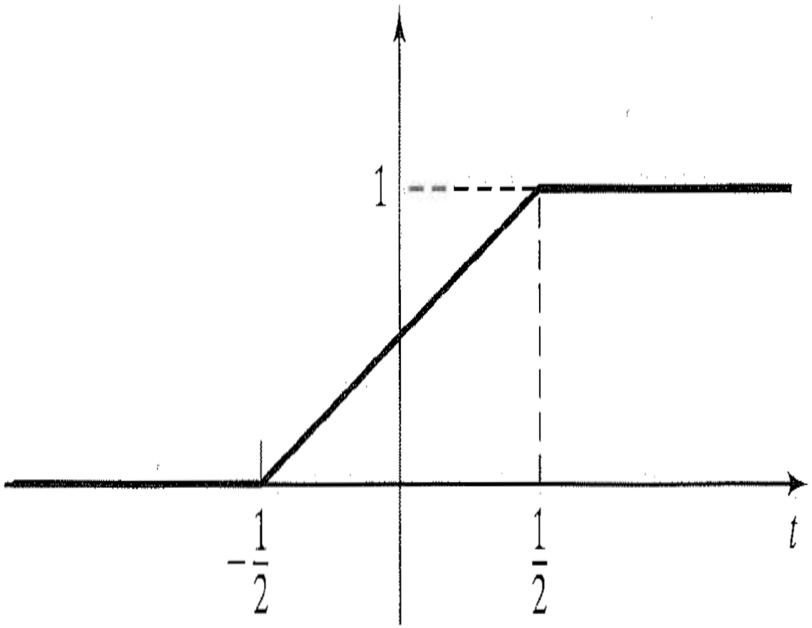

Figure 2.42 Signal $x(t)$.

Basic Properties of the Fourier Transform (31/38)

- **Example 2.3.19. (Cont'd)**
- Note that

$$x(t) = \int_{-\infty}^{t} \Pi(\tau)\, d\tau.$$

Using the integration theorem, we obtain

$$\mathscr{F}[x(t)] = \frac{\operatorname{sinc}(f)}{j2\pi f} + \frac{1}{2}\operatorname{sinc}(0)\delta(f)$$

$$= \frac{\operatorname{sinc}(f)}{j2\pi f} + \frac{1}{2}\delta(f)$$

Basic Properties of the Fourier Transform (32/38)

- **Moments.** If $\mathscr{F}[x(t)]=X(f)$, then the nth moment of $x(t)$ can be obtained from the relation

$$\int_{-\infty}^{\infty} t^n x(t)dt = \left(\frac{j}{2\pi}\right)^n \frac{d^n}{df^n} X(f)|_{f=0}$$

- This can be shown by using the differentiation in the frequency domain result. We have

$$\mathscr{F}[t^n x(t)] = \left(\frac{j}{2\pi}\right)^n \frac{d^n}{df^n} X(f)$$

This means that

$$\int_{-\infty}^{\infty} t^n x(t) e^{-j2\pi ft} dt = \left(\frac{j}{2\pi}\right)^n \frac{d^n}{df^n} X(f)$$

Letting $f=0$, we obtain the desired result

Basic Properties of the Fourier Transform (33/38)

- For the special case of n=0, we obtain this simple relation for finding the area under a signal, i.e.,

$$\int_{-\infty}^{\infty} x(t)dt = X(0)$$

Basic Properties of the Fourier Transform (34/38)

- **Example 2.3.20.** Determine the *n*th moment of $x(t)=e^{-\alpha t}u_{-1}(t)$, where $\alpha>0$
- First we solve for $X(f)$. We have

$$X(f)=\int_{-\infty}^{\infty} e^{-\alpha t}u_{-1}(t)e^{-j2\pi ft}dt$$

$$=\frac{1}{\alpha+j2\pi f}$$

By differentiating n times, we obtain

$$\frac{d^n}{df^n}X(f)=\frac{n!(-j2\pi)^n}{(\alpha+j2\pi f)^{n+1}}.$$

Hence, $\int_{-\infty}^{\infty} t^n e^{-\alpha t}u_{-1}(t)dt=\left(\frac{j}{2\pi}\right)^n n!(-j2\pi)^n\frac{1}{\alpha^{n+1}}=\frac{n!}{\alpha^{n+1}}$

Basic Properties of the Fourier Transform (35/38)

- **Example 2.3.21.** Determine the Fourier transform of $x(t)=e^{-\alpha|t|}$, where $\alpha>0$.

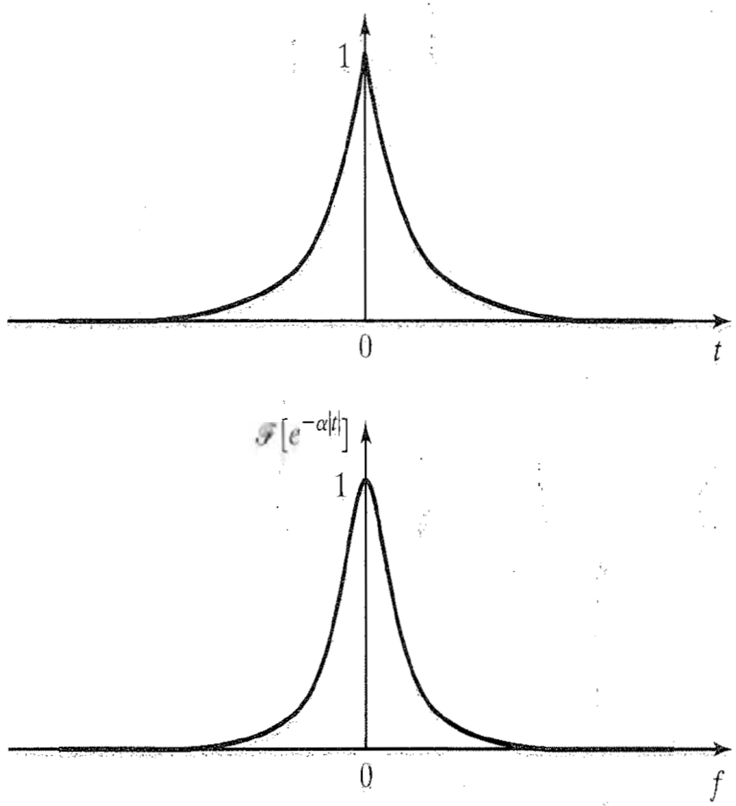

Figure 2.43 Signal $e^{-\alpha|t|}$ and its Fourier transform.

Basic Properties of the Fourier Transform (36/38)

- **Example 2.3.21.(Cont'd)** We have

$$x(t)=e^{-\alpha t}u_{-1}(t)+e^{\alpha t}u_{-1}(-t)=x_1(t)+x_2(t)$$

We already see that

$$\mathscr{F}[x_1(t)]=\mathscr{F}[e^{-\alpha t}u_{-1}(t)]=\frac{1}{\alpha+j2\pi f}$$

and

$$\mathscr{F}[x_2(-t)]=\frac{1}{\alpha-j2\pi f}$$

Hence by the linearity property, we have

$$\mathscr{F}[x(t)]=\frac{1}{\alpha+j2\pi f}+\frac{1}{\alpha-j2\pi f}$$

$$=\frac{2\alpha}{\alpha^2+4\pi^2 f^2}$$

Basic Properties of the Fourier Transform (37/38)

TABLE 2.1 TABLE OF FOURIER-TRANSFORM PAIRS

Time Domain	Frequency Domain
$\delta(t)$	1
1	$\delta(f)$
$\delta(t - t_0)$	$e^{-j2\pi f t_0}$
$e^{j2\pi f_0 t}$	$\delta(f - f_0)$
$\cos(2\pi f_0 t)$	$\frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$
$\sin(2\pi f_0 t)$	$-\frac{1}{2j}\delta(f + f_0) + \frac{1}{2j}\delta(f - f_0)$
$\Pi(t)$	$\text{sinc}(f)$
$\text{sinc}(t)$	$\Pi(f)$
$\Lambda(t)$	$\text{sinc}^2(f)$
$\text{sinc}^2(t)$	$\Lambda(f)$
$e^{-\alpha t}u_{-1}(t),\ \alpha > 0$	$\frac{1}{\alpha + j2\pi f}$
$te^{-\alpha t}u_{-1}(t),\ \alpha > 0$	$\frac{1}{(\alpha + j2\pi f)^2}$
$e^{-\alpha\|t\|}$	$\frac{2\alpha}{\alpha^2 + (2\pi f)^2}$
$e^{-\pi t^2}$	$e^{-\pi f^2}$
$\text{sgn}(t)$	$\frac{1}{j\pi f}$
$u_{-1}(t)$	$\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$
$\delta'(t)$	$j2\pi f$
$\delta^{(n)}(t)$	$(j2\pi f)^n$
$\frac{1}{t}$	$-j\pi\,\text{sgn}(f)$
$\sum_{n=-\infty}^{n=+\infty} \delta(t - nT_0)$	$\frac{1}{T_0}\sum_{n=-\infty}^{n=+\infty} \delta\left(f - \frac{n}{T_0}\right)$

Basic Properties of the Fourier Transform (38/38)

TABLE 2.2 TABLE OF FOURIER-TRANSFORM PROPERTIES

Signal	Fourier Transform
$\alpha x_1(t) + \beta x_2(t)$	$\alpha X_1(f) + \beta X_2(f)$
$X(t)$	$x(-f)$
$x(at)$	$\frac{1}{\|a\|} X\left(\frac{f}{a}\right)$
$x(t - t_0)$	$e^{-j2\pi f t_0} X(f)$
$e^{j2\pi f_0 t} x(t)$	$X(f - f_0)$
$x(t) \star y(t)$	$X(f)Y(f)$
$x(t)y(t)$	$X(f) \star Y(f)$
$\frac{d}{dt} x(t)$	$j2\pi f X(f)$
$\frac{d^n}{dt^n} x(t)$	$(j2\pi f)^n X(f)$
$t x(t)$	$\left(\frac{j}{2\pi}\right) \frac{d}{df} X(f)$
$t^n x(t)$	$\left(\frac{j}{2\pi}\right)^n \frac{d^n}{df^n} X(f)$
$\int_{-\infty}^{t} x(\tau)\, d\tau$	$\frac{X(f)}{j2\pi f} + \frac{1}{2} X(0)\delta(f)$

Fourier Transform for Periodic Signals (1/3)

- Let $x(t)$ be a periodic signal with the period T_0. Let $\{x_n\}$ denote the Fourier series coefficients corresponding to this signal. There exists another way to find $\{x_n\}$ through the Fourier transform of the truncated signal $x_{T_0}(t)$ as

$$x_{T_0}(t) = \begin{cases} x(t), & -\frac{T_0}{2} < t \le \frac{T_0}{2} \\ 0, & otherwise \end{cases}$$

- Rewrite $x(t)$ in terms of $x_{T_0}(t)$

$$x(t) = x_{T_0}(t) \star \sum_{n=-\infty}^{\infty} \delta(t - nT_0).$$

Taking the Fourier transform on both sides of $x(t)$, we obtain

$$X(f) = X_{T_0}(f)\left[\frac{1}{T_0}\sum_{n=-\infty}^{\infty}\delta(f - \tfrac{n}{T_0})\right]$$

Fourier Transform for Periodic Signals (2/3)

- $X(f)$ can be further rewritten as

$$X(f) = \frac{1}{T_0}\left[\sum_{n=-\infty}^{\infty} X_{T_0}(\tfrac{n}{T_0})\delta(f - \tfrac{n}{T_0})\right].$$

- Consider the Fourier series of $x(t)$. We have

$$x(t) = \sum_{n=-\infty}^{\infty} x_n e^{j2\pi \frac{n}{T_0} t}.$$

Take Fourier transform on both sides of $x(t)$. We obtain

$$X(f) = \sum_{n=-\infty}^{\infty} x_n \delta(f - \tfrac{n}{T_0}).$$

We thus conclude

$$x_n = \frac{1}{T_0} X_{T_0}\left(\frac{n}{T_0}\right) \qquad \text{(Eq. 2.3.64)}$$

Fourier Transform for Periodic Signals (3/3)

- Given the periodic signal $x(t)$, we can find x_n by using the following steps:
  - First, we determine the truncated signal $x_{T_0}(t)$
  - Then, we determine the Fourier transform of the truncated signal using Table 2.1 and the Fourier-transform properties
  - Finally, we evaluate the Fourier transform of the truncated signal at $f = n/T_0$ and scale it by $1/T_0$, as shown in Eq. (2.3.64)

Transmission over LTI Systems (1/7)

- Let $X(f)$, $H(f)$, and $Y(f)$ be the Fourier transforms of the input, system impulse response, and the output, respectively. Thus,

$$Y(f)=H(f)X(f)$$

- **Example 2.3.23.** Let the input to an LTI system be the signal

$$x(t)=\mathrm{sinc}(W_1 t)$$

and let the impulse response of the system be

$$h(t)=\mathrm{sinc}(W_2 t).$$

Determine the output signal.

Transmission over LTI Systems (2/7)

- **Example 2.3.23. (Cont'd)** First, we transform the signals to the frequency domain. Thus, we obtain

$$X(f) = \frac{1}{W_1}\Pi(\frac{f}{W_1})$$

and

$$H(f) = \frac{1}{W_2}\Pi(\frac{f}{W_2})$$

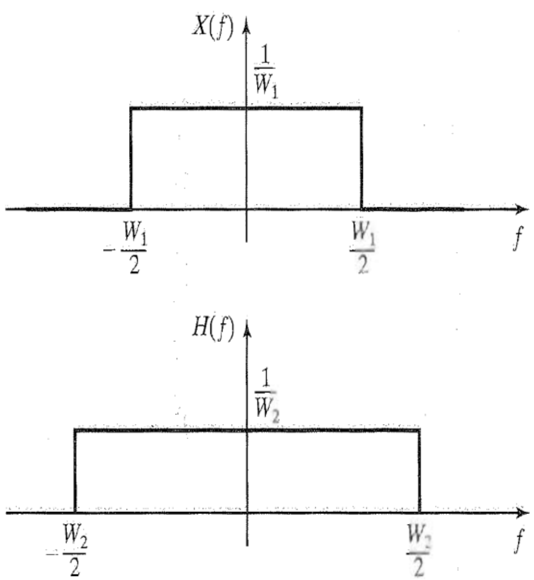

Figure 2.44 Lowpass signal and lowpass filter.

Transmission over LTI Systems (3/7)

- **Example 2.3.23. (Cont'd)** To obtain the output in the frequency domain, we have

$$
\begin{aligned}
Y(f) &= X(f)H(f) \\
&= \frac{1}{W_1W_2}\Pi\left(\frac{f}{W_1}\right)\Pi\left(\frac{f}{W_2}\right) \\
&= \begin{cases} \frac{1}{W_1W_2}\Pi\left(\frac{f}{W_1}\right), & W_1 \le W_2 \\ \frac{1}{W_1W_2}\Pi\left(\frac{f}{W_2}\right), & W_1 > W_2 \end{cases}
\end{aligned}
$$

From this result, we obtain

$$
y(t) = \begin{cases} \frac{1}{W_2}\operatorname{sinc}(W_1 t), & W_1 \le W_2 \\ \frac{1}{W_1}\operatorname{sinc}(W_2 t), & W_1 > W_2 \end{cases}
$$

Transmission over LTI Systems (4/7)

- The bandwidth of a filter is the set of positive frequencies that a filter can pass

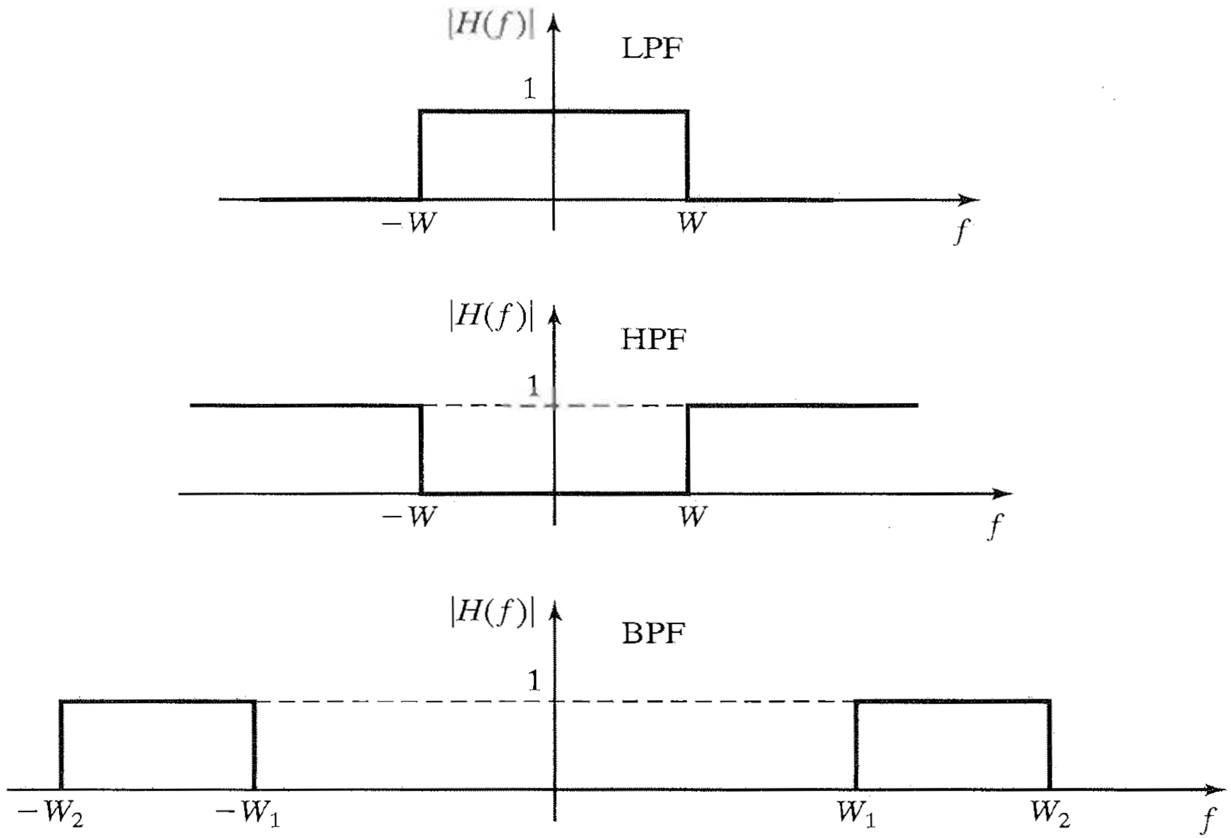

Figure 2.45 Various filter types.

Transmission over LTI Systems (5/7)

- For nonideal lowpass or bandpass filters, the bandwidth is usually defined as the band of frequencies at which the power-transfer ratio of the filter is half of the maximum power-transfer ratio
- This bandwidth is usually called the 3-dB bandwidth of the filter, because reducing the power by a factor of two is equivalent to decreasing it by 3 dB on the logarithmic scale

Transmission over LTI Systems (6/7)

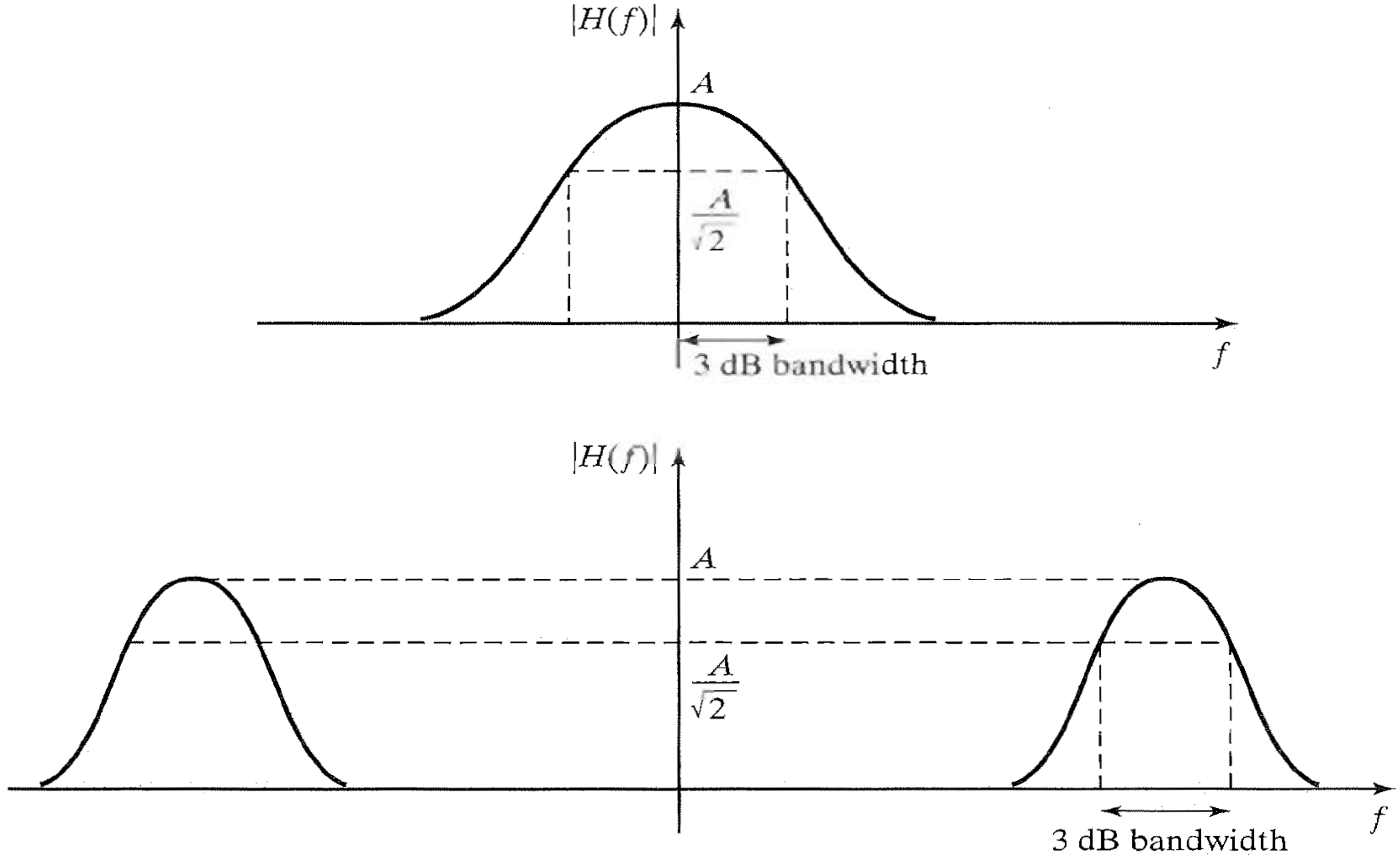

Figure 2.46 3 dB bandwidth of filters in Example 2.3.24.

Transmission over LTI Systems (7/7)

- **Example 2.3.24.** The magnitude of the transfer function of a filter is given by

$$H(f) = \frac{1}{\sqrt{1+\left(\frac{f}{10000}\right)^2}}.$$

Determine the filter type and its 3 dB bandwidth.

- This is a lowpass filter. A 3-dB bandwidth is10 kHz

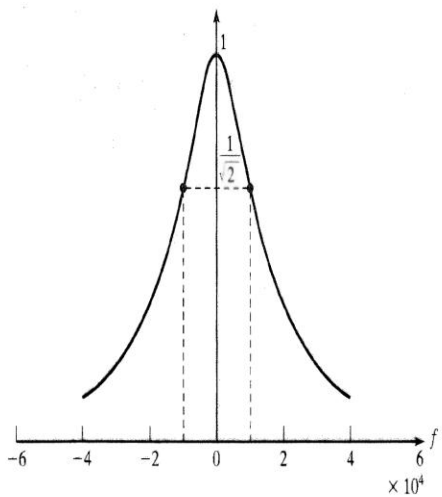

Figure 2.47 3 dB bandwidth of filter in Example 2.3.24.