Chapter 2 Signals and Linear Systems (III)

Parseval's Relation (1/3)

- Parseval's relation says that the power of a periodic signal is the sum of the power contents of its components in the Fourier-series representation of that signal
- Or, equivalently, the power content of the periodic signal is the sum of the power contents of harmonics
- Let the Fourier-series representation of the periodic signal $x(t)$ is given by

$$x(t) = \sum_{n=-\infty}^{+\infty} x_n e^{j2\pi \frac{n}{T_0} t}$$

The complex conjugate of $x(t)$ is

$$x^*(t) = \sum_{n=-\infty}^{+\infty} x_n^* e^{-j2\pi \frac{n}{T_0} t}$$

Parseval's Relation (2/3)

- By multiplying the two equations, we have

$$|x(t)|^2 = \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{\infty} x_n x_m^* e^{j2\pi \frac{n-m}{T_0} t}$$

- We integrate both sides over one period and note

$$\int_{\alpha}^{\alpha+T_0} e^{j2\pi \frac{n-m}{T_0} t} dt = T_0 \delta_{mn} = \begin{cases} T_0, & n = m \\ 0, & n \neq m \end{cases}$$

Thus,

$$\int_{\alpha}^{\alpha+T_0} |x(t)|^2 \, dt = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x_n x_m^* T_0 \delta_{mn}$$

$$= T_0 \sum_{n=-\infty}^{\infty} |x_n|^2$$

Parseval's Relation (3/3)

- Finally, we have

$$\frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} |x(t)|^2 dt = \sum_{n=-\infty}^{\infty} |x_n|^2$$

Fourier Transform – From Fourier Series to Fourier Transforms (1/10)

- If we apply Fourier transform to a nonperiodic signal, the resulting spectrum will no longer be discrete
- The Fourier transform (or Fourier integral) of $x(t)$ is defined by

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$

- The original signal can be obtained from its Fourier transform by

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$

Fourier Transform – From Fourier Series to Fourier Transforms (2/10)

- The sufficient conditions for $x(t)$ to have a Fourier transform are Dirichlet conditions:

1. $x(t)$ is absolutely integrable on the real line; that is

$$\int_{-\infty}^{\infty} |x(t)|dt < \infty$$

2. The number of maxima and minima of $x(t)$ in any finite interval on the real line is finite
3. The number of discontinuities of $x(t)$ in any finite interval on the real line is finite

Fourier Transform – From Fourier Series to Fourier Transforms (3/10)

- $X(f)$ is generally a complex function. Its magnitude $|X(f)|$ and phase $\angle X(f)$ represent the amplitude and phase of various frequency components in $x(t)$
- We employ the following notation:

$$X(f) = \mathscr{F}[x(t)]$$

and

$$x(t) = \mathscr{F}^{-1}[X(f)].$$

Sometimes we use a shorthand for both relations:

$$x(t) \Leftrightarrow X(f)$$

Fourier Transform – From Fourier Series to Fourier Transforms (4/10)

- If the variable in the Fourier transform is ω rather than f, then we have

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

and

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{-j\omega t} d\omega$$

- Since

$$\int_{-\infty}^{\infty} \delta(t) e^{-j2\pi ft} dt = 1,$$

we have

$$\delta(t) = \int_{-\infty}^{\infty} e^{j2\pi ft} df$$

and

$$\delta(t-\tau) = \int_{-\infty}^{\infty} e^{j2\pi f(t-\tau)} df.$$

Fourier Transform – From Fourier Series to Fourier Transforms (5/10)

- **Example 2.3.1.** Find the Fourier transform of $\Pi(t)$.

We have

$$\begin{aligned}
\mathscr{F}[x(t)] &= \int_{-\infty}^{+\infty} \Pi(t) e^{-j2\pi f t} dt \\
&= \int_{-\frac{1}{2}}^{+\frac{1}{2}} e^{-j2\pi f t} dt \\
&= \frac{1}{-j2\pi f}\left[e^{-j\pi f} - e^{j\pi f}\right] \\
&= \frac{\sin(\pi f)}{\pi f} \\
&= \operatorname{sinc}(f).
\end{aligned}$$

Therefore

$$\mathscr{F}[\Pi(t)] = \operatorname{sinc}(f)$$

Fourier Transform – From Fourier Series to Fourier Transforms (6/10)

- **Example 2.3.1(Cont'd)**

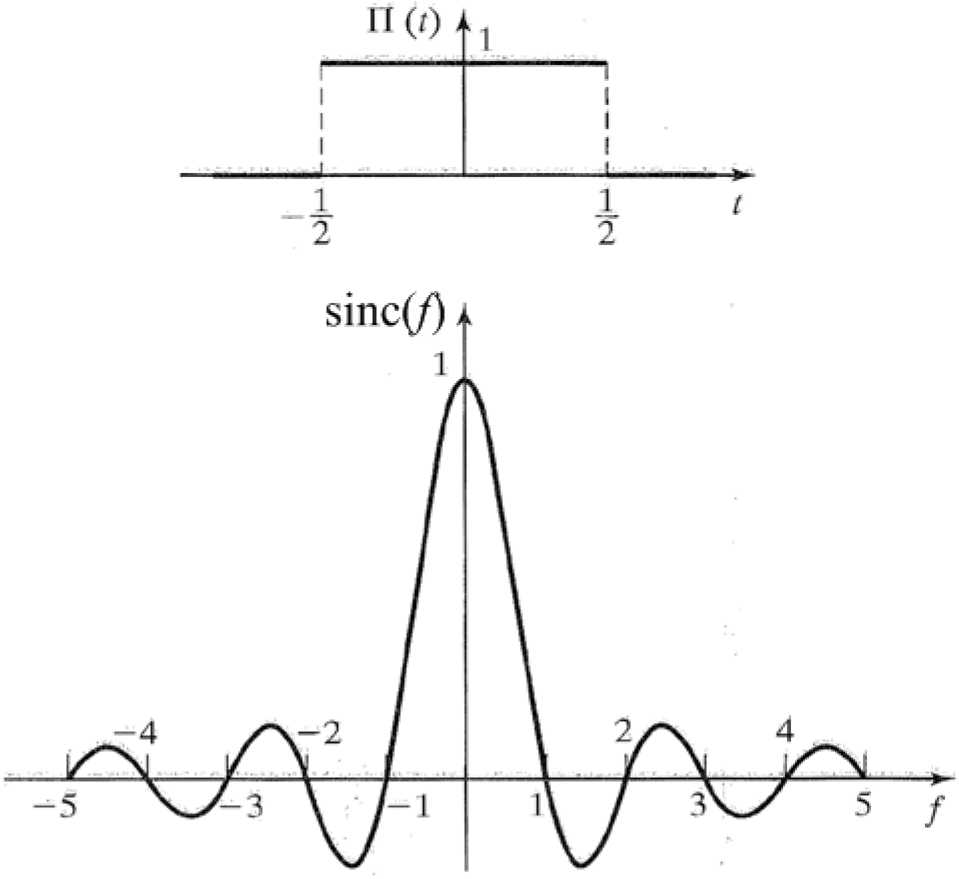

Figure 2.33 Π(t) and its Fourier transform.

Fourier Transform – From Fourier Series to Fourier Transforms (7/10)

- **Example 2.3.2.** Determine the Fourier transform of an impulse signal $x(t)=\delta(t)$.

$$\mathscr{F}[\delta(t)]=1$$

$$\mathscr{F}^{-1}[1]=\delta(t)$$

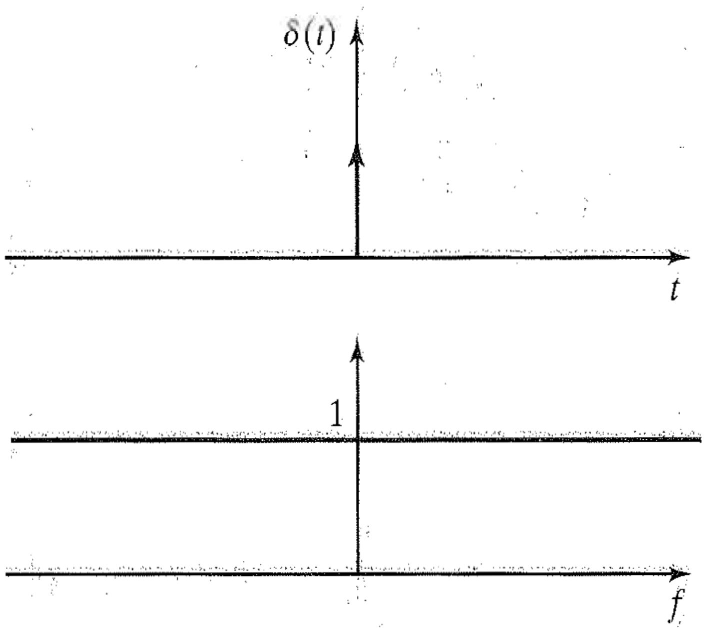

Figure 2.34 Impulse signal and its spectrum.

Fourier Transform – From Fourier Series to Fourier Transforms (8/10)

- **Example 2.3.3.** Determine the Fourier transform of signal sgn(t).

We begin with the definition of sgn(t) as a limit of an exponential function and given by

$$x_n(t) = \begin{cases} e^{-\frac{t}{n}} & ,t > 0 \\ -e^{\frac{t}{n}} & ,t < 0 \\ 0 & ,t = 0. \end{cases}$$

For this signal, the Fourier transform is

$$X_n(f) = \frac{-j4\pi f}{\frac{1}{n^2} + 4\pi^2 f^2}$$

Fourier Transform – From Fourier Series to Fourier Transforms (9/10)

- **Example 2.3.3. (Cont'd)** Now, letting $n \to \infty$, we obtain

$$\mathscr{F}[\operatorname{sgn}(t)] = \lim_{n\to\infty} X_n(f)$$
$$= \lim_{n\to\infty} \frac{-j4\pi f}{\frac{1}{n^2} + 4\pi^2 f^2}$$
$$= \frac{1}{j\pi f}$$

Fourier Transform – From Fourier Series to Fourier Transforms (10/10)

- **Example 2.3.3. (Cont'd)**

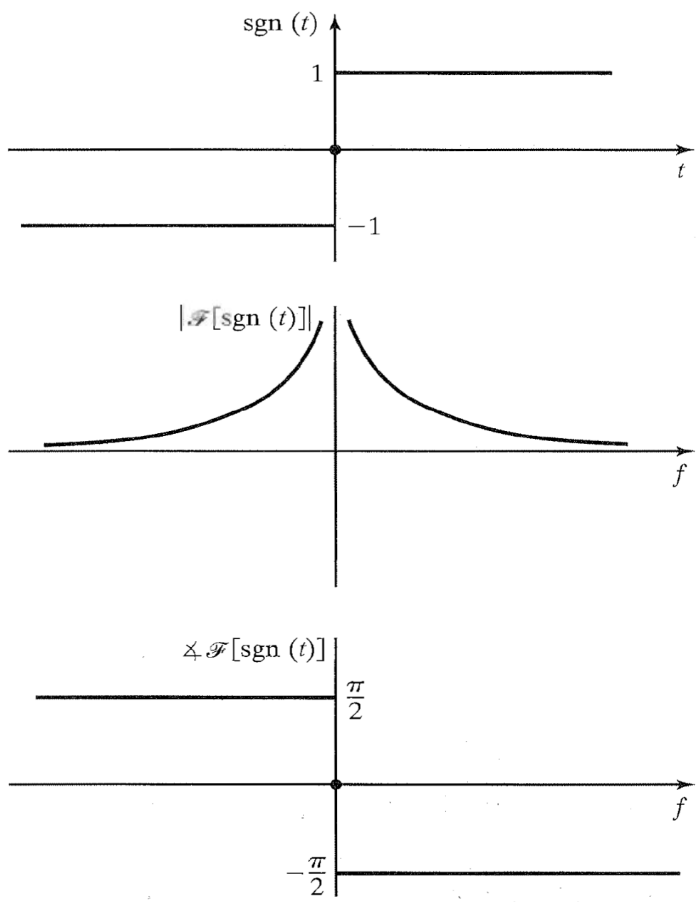

Figure 2.35 The signum signal and its spectrum.

Fourier Transform – Fourier Transform of Real, Even, and Odd Signals (1/3)

- The Fourier transform relation can be generally written as

$$X(f)=\int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$$
$$=\int_{-\infty}^{\infty} x(t)\cos(2\pi ft)dt - j\int_{-\infty}^{\infty} x(t)\sin(2\pi ft)dt$$

- If $x(t)$ is real, we have

$$x(t)=x_e(t)+x_o(t).$$

$X(f)$ can be rewritten as

$$X(f)=\int_{-\infty}^{\infty} x_e(t)\cos(2\pi ft)dt + \int_{-\infty}^{\infty} x_o(t)\cos(2\pi ft)dt$$
$$- j\int_{-\infty}^{\infty} x_e(t)\sin(2\pi ft)dt - j\int_{-\infty}^{\infty} x_o(t)\sin(2\pi ft)dt.$$

Term 2 and term 3 are zero. $X(f)$ is reduced to

$$X(f)=\int_{-\infty}^{\infty} x_e(t)\cos(2\pi ft)dt - j\int_{-\infty}^{\infty} x_o(t)\sin(2\pi ft)dt$$

Fourier Transform – Fourier Transform of Real, Even, and Odd Signals (2/3)

- We also note that the real part of $X(f)$ is an even function of the variable f. Likewise, the imaginary part of $X(f)$ is an odd function of the variable f

  ➔ $\mathrm{Re}[X(f)]=\mathrm{Re}[X(-f)]$, even

  $\mathrm{Im}[X(f)]=-\mathrm{Im}[X(-f)]$, odd

  ➔ $X(-f)=X^*(f)$, Hermitian

We also have

➔ $|X(-f)|=|X(f)|$, even

$\angle X(-f)=-\angle X(f)$, odd

Fourier Transform – Fourier Transform of Real, Even, and Odd Signals (3/3)

- If $x(t)$ is real and even, $X(f)$ can further be reduced to

$$X(f) = \int_{-\infty}^{\infty} x_e(t)\cos(2\pi f t)\,dt$$

➔ $X(f)$ real, even

- If $x(t)$ is real and odd, $X(f)$ is now be reduced to

$$X(f) = -j\int_{-\infty}^{\infty} x_o(t)\sin(2\pi f t)\,dt$$

➔ $X(f)$ imaginary, odd

Fourier Transform – Signal Bandwidth (1/1)

- We define the bandwidth of a real signal $x(t)$ as the range of positive frequencies present in the signal
- In order to find the bandwidth of $x(t)$, we first find $X(f)$, which is the Fourier transform of $x(t)$; then, we find the range of positive frequencies that $X(f)$ occupies
- The bandwidth is $BW=W_{max}-W_{min}$, where W_{max} is the highest positive frequency present in $X(f)$ and W_{min} is the lowest positive frequency present in $X(f)$