Chapter 2 Signals and Linear Systems (II)

Classification of Systems (1/12)

- A system is an interconnection of various elements or devices that, from a certain viewpoint, behave as a whole
- The most important point in the definition of a system is that *its output must be uniquely defined for any legitimate input*
- This definition can be written as

$$y(t)=\mathscr{F}[x(t)]$$

where $x(t)$ is the input, $y(t)$ is the output, $\mathscr{F}$ is the operation performed by the system

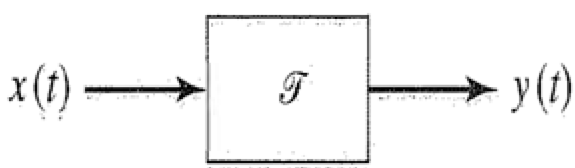

Figure 2.21 A system with an input and output.

Classification of Systems (2/12)

- **Example 2.1.16.** The input-output relationship $y(t)=3x(t)+3x^2(t)$ defines a system. For any input $x(t)$, the output $y(t)$ is uniquely determined
- A system is defined by two characteristics: 1) the operation that describes the system and 2) the set of legitimate input signals
- The operator $\mathscr{F}$ denotes the operation that describes the system
- $\mathscr{X}$ denotes the space of legitimate input to the system

Classification of Systems (3/12)

- **Example 2.1.17.** The system described by the input-output relationship

$$y(t) = \mathscr{F}[x(t)] = \frac{d}{dt}x(t)$$

for which $\mathscr{X}$ is the space of all differentiable signals, describes the system. This system is referred to as the *differentiator*

Classification of Systems – Discrete-Time and Continuous-Time Systems (4/12)

- A *discrete-time system* accepts discrete-time signals as the input and produces discrete-time signal at the output
- A *continuous-time system* accepts continuous-time signals as the input and produces continuous-time signal at the output
- **Example 2.1.18.** The system described by

$$y[n]=x[n]-x[n-1]$$

is a *discrete-time differentiator*

Classification of Systems – Linear and Nonlinear Systems (5/12)

- Linear systems are systems for which the *superposition* property is satisfied, i.e., the system's response to a linear combination of the inputs is the linear combination of the responses to the corresponding inputs
- A system $\mathscr{F}$ is linear if and only if, for any two input signals $x_1(t)$ and $x_2(t)$ and for any two scalars α and β, we have

$$\mathscr{F}[\alpha x_1(t)+\beta x_2(t)]=\alpha\mathscr{F}[x_1(t)]+\beta\mathscr{F}[x_2(t)]$$

A system that does not satisfy this relationship is called *nonlinear*

Classification of Systems – Linear and Nonlinear Systems (6/12)

- Linearity can also be defined in terms of the following two properties:

$$\mathscr{F}[x_1(t)+x_2(t)]=\mathscr{F}[x_1(t)]+\mathscr{F}[x_2(t)]$$

$$\mathscr{F}[\alpha x(t)]=\alpha\mathscr{F}[x(t)]$$

- A system that satisfies the first property is called additive, and a system that satisfies the second property is called homogeneous
- From the second property, we have $\mathscr{F}[0]=0$ in a linear system. In other words, the response of a linear system to a zero input is always zero (for linearity, this is a necessary condition but not a sufficient condition)

Classification of Systems – Linear and Nonlinear Systems (7/12)

- In a linear system, we can decompose the input into a linear combination of some fundamental signals whose output can be derived easily
- We denote the operation of linear systems by $\mathscr{L}$, rather than $\mathscr{F}$
- **Example 2.1.19.** The differentiator is a linear system. The system described by $y(t)=x^2(t)$ is nonlinear because

$$\mathscr{F}[2x(t)]=4x^2(t)\neq 2x^2(t)=2\mathscr{F}[x(t)]$$

- **Example 2.1.20.** A delay system defined by $y(t)=x(t-\Delta)$ is linear

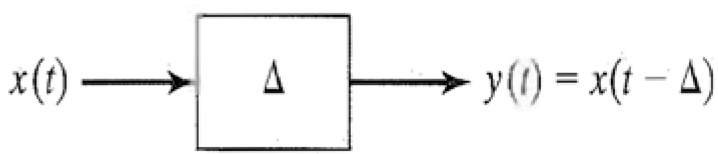

Figure 2.22 The input–output relation for the delay system.

Classification of Systems – Time-Invariant and Time-Varying Systems (8/12)

- A system is called *time-invariant* if its input-output relationship does not change with time. This means that a delayed version of an input results in a delayed version of the output
- A system is *time-invariant* if and only if, for all $x(t)$ and all values of t_0, its response to $x(t-t_0)$ is $y(t-t_0)$, where $y(t)$ is the response of the system to $x(t)$

Classification of Systems – Time-Invariant and Time-Varying Systems (9/12)

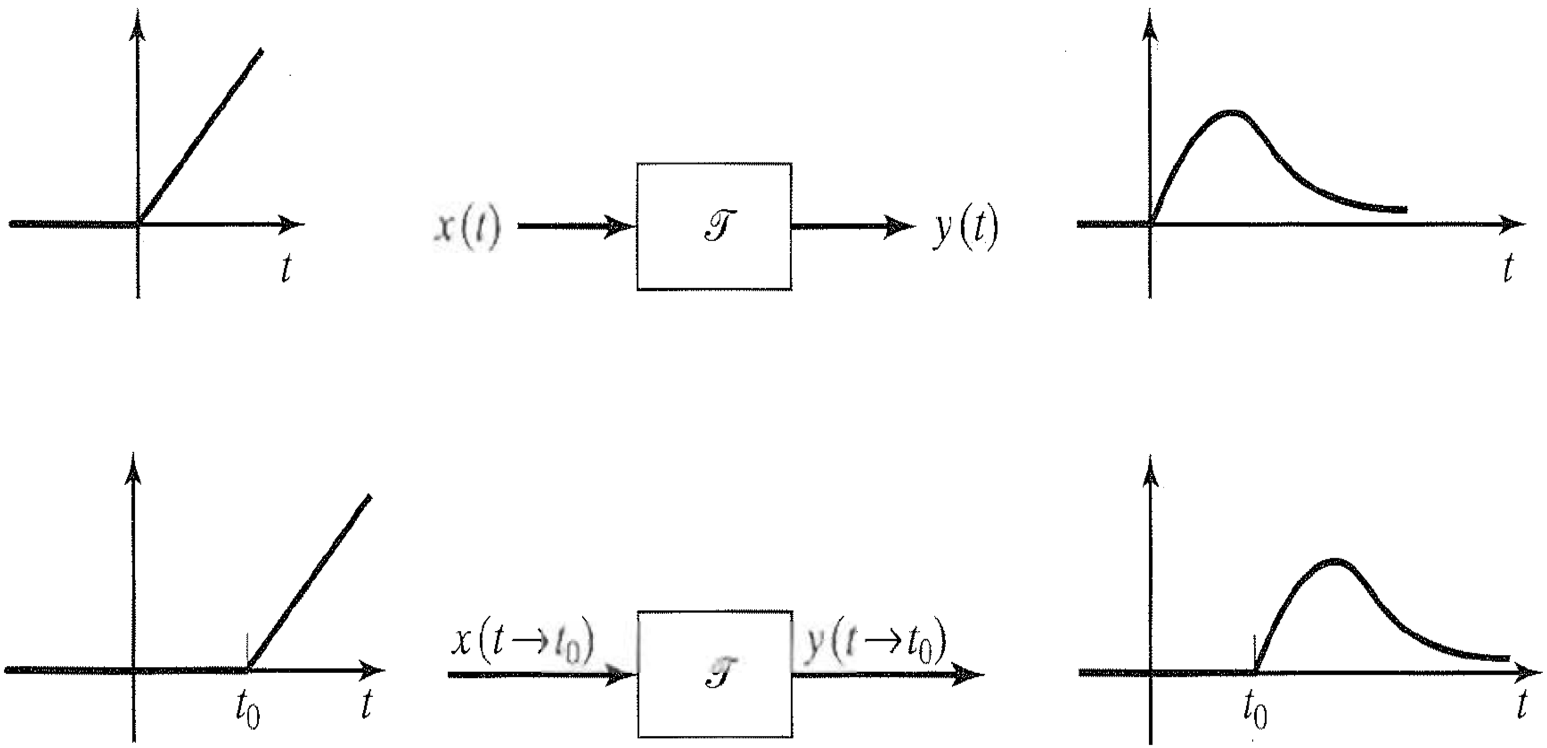

Figure 2.23 A time-invariant system.

Classification of Systems – Time-Invariant and Time-Varying Systems (10/12)

- **Example 2.1.21.** The differentiator is a time-invariant system
- **Example 2.1.22.** The modulator, defined by $y(t)=x(t)\cos(2\pi f_0 t)$ is an example of a time-varying system. The response of this system to $x(t-t_0)$ is

$$x(t-t_0)\cos(2\pi f_0 t),$$

  which is not equal to $y(t-t_0)$
- The class of *linear time-invariant* (LTI) system is particularly important. The response of these systems to inputs can be derived simply by finding the convolution of the input and the impulse response of the system

Classification of Systems – Causal and Noncausal Systems (11/12)

- In a physically realizable system, the output at any time depends only on the values of the input signal up to that time and does not depend on the future values of the input
- A system is *causal* if its output at any time t_0 depends on the input at times prior to t_0, i.e.,

$$y(t_0)=\mathscr{F}[x(t) : t \leq t_0]$$

- A necessary and sufficiency condition for an LTI system to be causal is that its impulse response $h(t)$ must be a causal signal, i.e., for $t<0$, we must have $h(t)=0$
- Noncausal systems are encountered in situation where signals are not processed in real time

Classification of Systems – Causal and Noncausal Systems (12/12)

- A differentiator is an example of a causal system since it is LTI and its impulse response $h(t)=\delta'(t)$, is zero for $t<0$
- A modulator is a causal but time-varying system
- The delay system is causal for $\Delta \geqq 0$ and noncausal for $\Delta<0$, since its impulse response $\delta(t-\Delta)$ is zero for $t<0$, if $\Delta>0$ and nonzero if $\Delta<0$

Analysis of LTI Systems in the Time Domain (1/4)

- The impulse response $h(t)$ of a system is the response of the system to a unit impulse input $\delta(t)$:

$$h(t) = \mathscr{L}[\delta(t)]$$

- The output $y(t)$ of an LTI system to any input signal $x(t)$ can be expressed by the convolution of $h(t)$ and $x(t)$

$$\begin{aligned} y(t) &= \mathscr{L}[x(t)] \\ &= \mathscr{L}\left[\int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau\right] \\ &= \int_{-\infty}^{+\infty} x(\tau)\mathscr{L}[\delta(t-\tau)]d\tau \\ &= \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau \\ &= x(t) \bigstar h(t) \end{aligned}$$

Analysis of LTI Systems in the Time Domain (2/4)

- **Example 2.1.24.** The system defined by

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

is called an integrator. Since integration is linear and the response to $x(t\text{-}t_0)$ is

$$y_1(t) = \int_{-\infty}^{t-t_0} x(u)du$$

$$= y(t - t_0)$$

- An integrator is LTI

Analysis of LTI Systems in the Time Domain (3/4)

- **Example 2.1.25.** Let a linear time-invariant system have the impulse response $h(t)$. Assume this system has a complex exponential signal as input, i.e., $x(t) = Ae^{j(2\pi f_0 t+\theta)}$. The response to the input can be obtained by

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau) A e^{j(2\pi f_0 (t-\tau)+\theta)} d\tau \\ &= Ae^{j\theta} e^{j2\pi f_0 t} \int_{-\infty}^{\infty} h(\tau) e^{-j2\pi f_0 \tau} d\tau \\ &= A|H(f_0)| e^{j(2\pi f_0 t+\theta+\angle H(f_0))} \end{aligned}$$

where

$$H(f_0) = |H(f_0)| e^{j\angle H(f_0)} = \int_{-\infty}^{\infty} h(\tau) e^{-j2\pi f_0 \tau} d\tau$$

Analysis of LTI Systems in the Time Domain (4/4)

- The response of an LTI system to the complex exponential with frequency f_0 is a complex exponential with the same frequency. The amplitude of the response can be obtained by multiplying the amplitude of the input by $|H(f_0)|$, and its phase is obtained by adding $\angle H(f_0)$ to the input phase
- Complex exponentials are called *eigenfunctions* of the class of linear time-invariant systems. The eigenfunctions of a system are the set of inputs for which the output is a scaling of the input
- Finding the response of LTI systems to the class of complex exponential signals is particularly simple

Fourier Series and Its Properties (1/17)

- The basic idea to find the response of an LTI system is *to expand the input as a linear combination of some basic signals whose output can be easily obtained, and then to employ the linearity properties of the system to obtain the corresponding output*
- The set of complex exponentials are the eigenfunctions of LTI systems
- The response of an LTI system to a complex exponential is a complex exponential with the same frequency *with a change in amplitude and phase*

Fourier Series and Its Properties (2/17)

- A periodic signal $x(t)$ with period T_0 can be expanded into its Fourier series if it meets the following Dirichlet conditions:

1. $x(t)$ is absolutely integrable over its period, i.e.,

$$\int_0^{T_0} |x(t)| < \infty$$

2. The number of maxima and minima of $x(t)$ in each period is finite
3. The number of discontinuities of $x(t)$ in each period is finite

- The Fourier series of $x(t)$ is $x(t) = \sum_{n=-\infty}^{\infty} x_n e^{j2\pi \frac{n}{T_0} t}$

  where

$$x_n = \frac{1}{T_0} \int_{\alpha}^{\alpha+T_0} x(t) e^{-j2\pi \frac{n}{T_0} t} dt$$

Fourier Series and Its Properties (3/17)

- The Dirichlet conditions are only *sufficient* conditions for the existence of the Fourier series expansion. For some signals that do not satisfy these conditions, we can still find the Fourier-series expansion
- The quantity $f_0 = \frac{1}{T_0}$ is called the fundamental frequency of the signal $x(t)$. The nth multiple of f_0 is called the nth harmonic

Fourier Series and Its Properties (4/17)

- An example *discrete spectrum* of the periodic signal $x(t)$

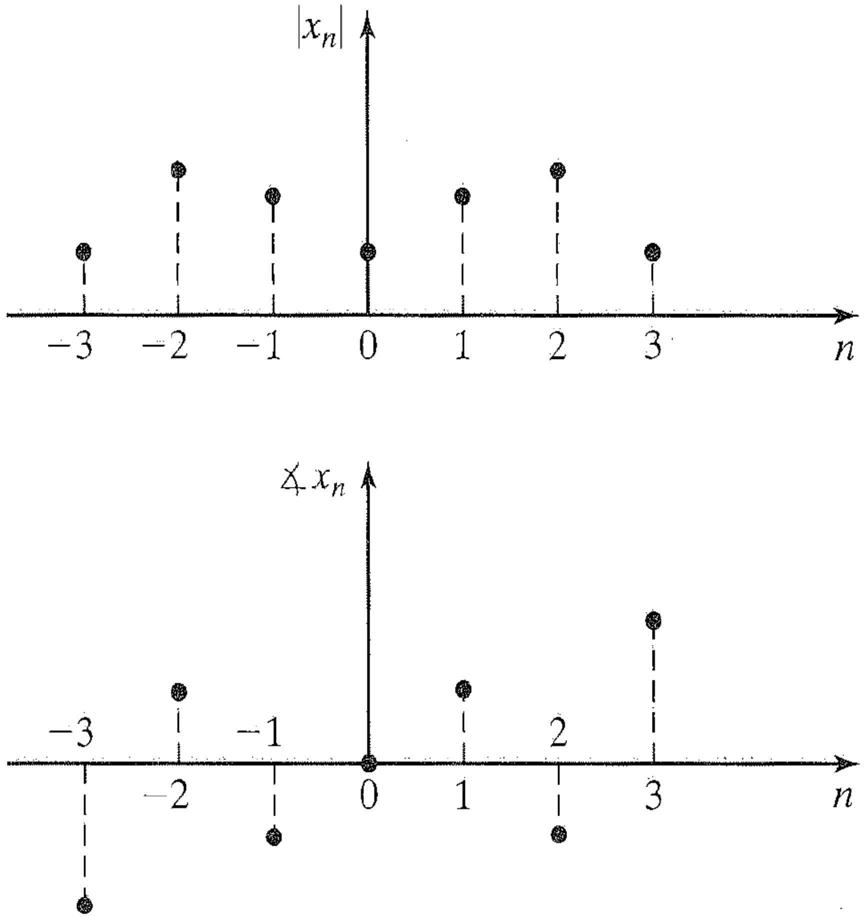

Figure 2.24 The discrete spectrum of $x(t)$.

Fourier Series and Its Properties (5/17)

- **Example 2.2.1.** Let $x(t)$ is defined by

$$x(t) = \sum_{n=-\infty}^{\infty} \Pi\left(\frac{t - nT_0}{\tau}\right).$$

Find its Fourier-series expansion.

The Fourier-series expansion is

$$x(t) = \sum_{n=-\infty}^{\infty} \frac{\tau}{T_0} \operatorname{sinc}\left(\frac{\tau n}{T_0}\right) e^{j2\pi n \frac{1}{T_0} t}$$

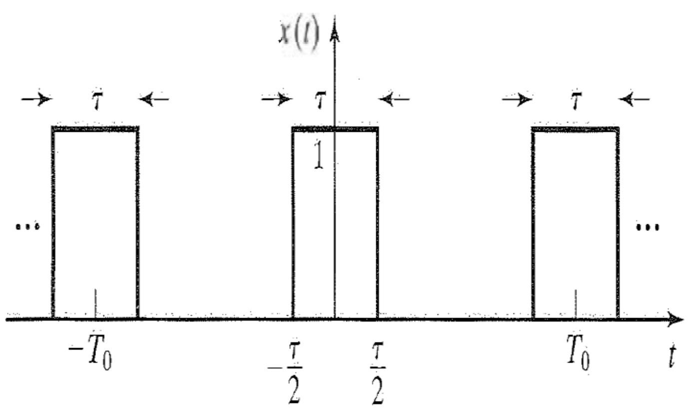

Figure 2.25 Periodic signal $x(t)$ in Equation (2.2.6).

Fourier Series and Its Properties (6/17)

- **Example 2.2.1. (Cont'd)**

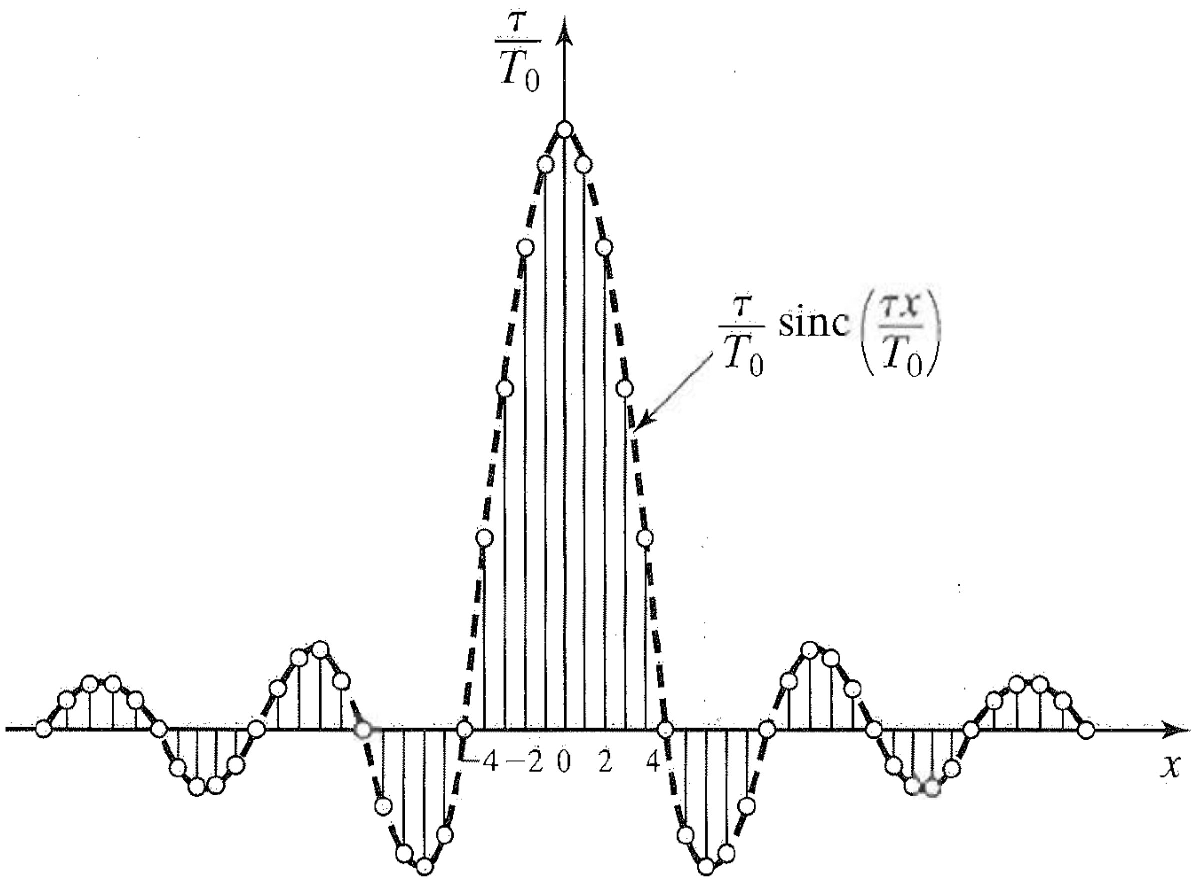

Figure 2.26 The discrete spectrum of the rectangular-pulse train.

Fourier Series and Its Properties (7/17)

- **Example 2.2.2.** Determine the Fourier-series expansion for the signal $x(t)$ described by

$$x(t) = \sum_{n=-\infty}^{\infty} (-1)^n \Pi(t-n)$$

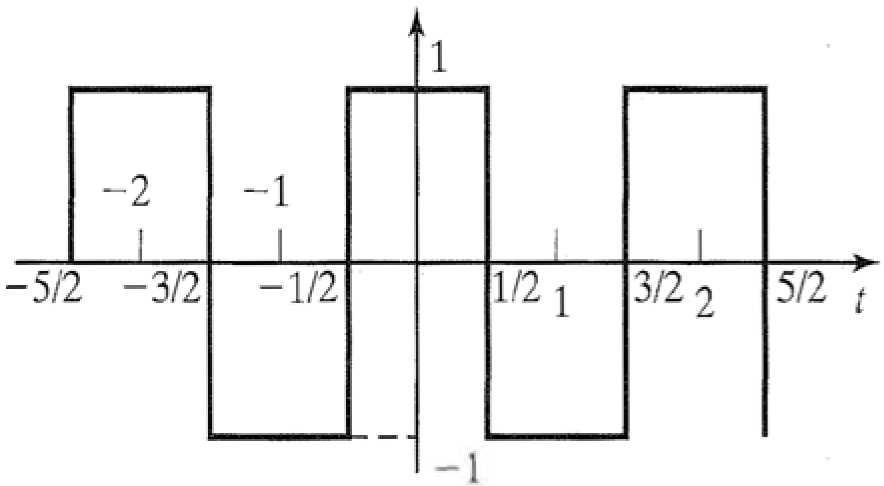

Figure 2.27 Signal $x(t)$ in Equation (2.2.9)

$$x(t) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \cos[(2k+1)\pi t]$$

Fourier Series and Its Properties (8/17)

- **Example 2.2.3.** Determine the Fourier-series representation of an *impulse train* denoted by

$$x(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT_0)$$

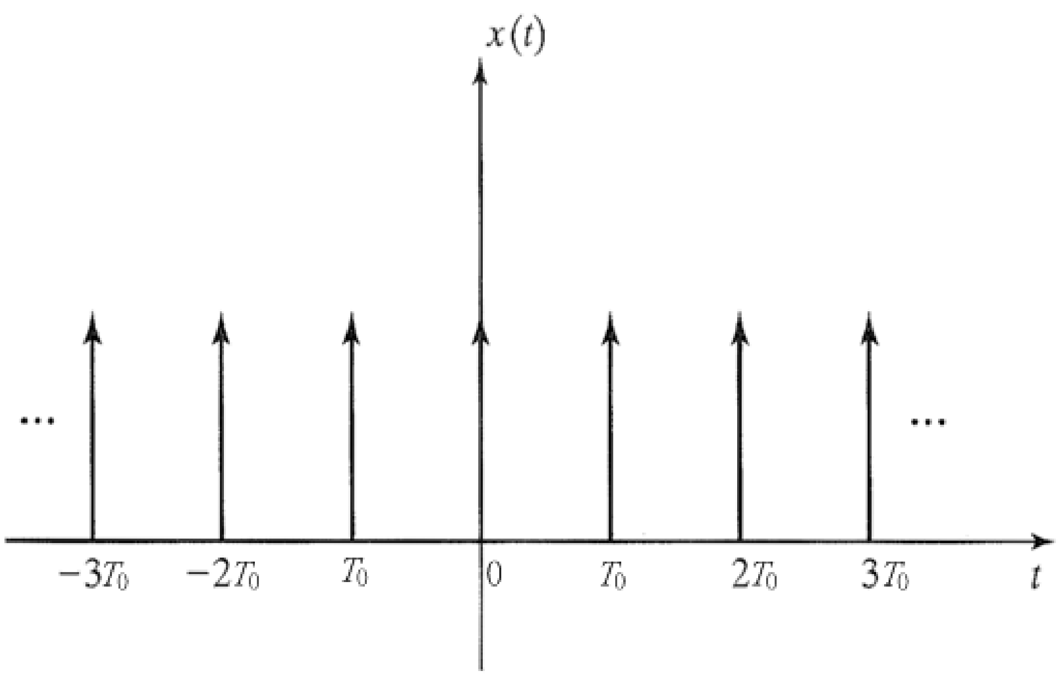

Figure 2.28 An impulse train.

Fourier Series and Its Properties (9/17)

- **Example 2.2.3. (Cont'd)**

We have

$$x_n = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)e^{-j2\pi\frac{n}{T_0}t}dt$$

$$= \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \delta(t)e^{-j2\pi\frac{n}{T_0}t}dt$$

$$= \frac{1}{T_0}$$

With these coefficients, we have the following expansion:

$$\sum_{n=-\infty}^{\infty}\delta(t-nT_0) = \frac{1}{T_0}\sum_{n=-\infty}^{\infty}e^{j2\pi\frac{n}{T_0}t}$$

Fourier Series and Its Properties – Positive and Negative Frequencies (10/17)

- The Fourier-series expansion of a periodic signal $x(t)$ is expressed as

$$x(t) = \sum_{n=-\infty}^{\infty} x_n e^{j2\pi \frac{n}{T_0} t},$$

in which all positive and negative multiples of the fundamental frequency $1/T_0$ are presented.

- A positive frequency corresponds to a term of the form $e^{j\omega t}$. A negative frequency corresponds to a term of the form $e^{-j\omega t}$

Fourier Series and Its Properties – Positive and Negative Frequencies (11/17)

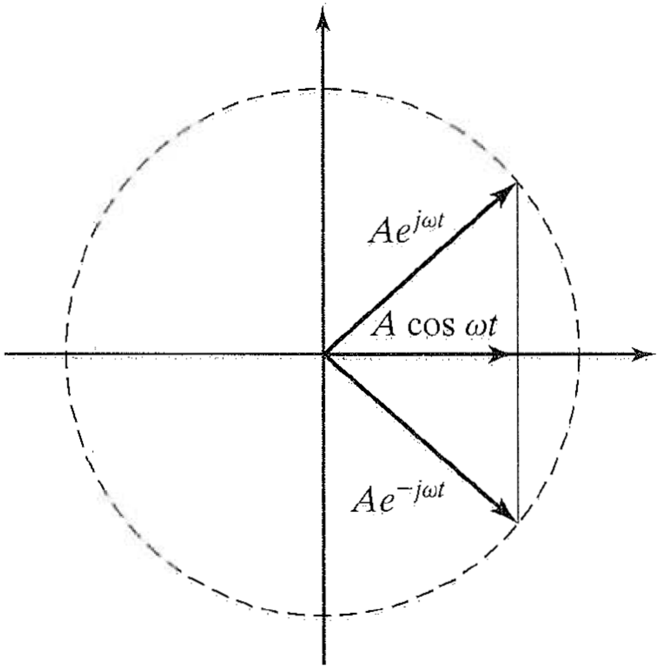

Figure 2.29 Phasors representing positive and negative frequencies.

Fourier Series and Its Properties – Real Signals (12/17)

- For real $x(t)$, we have

$$x_{-n} = \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)e^{j2\pi\frac{n}{T_0}t}dt$$

$$= \left[\frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)e^{-j2\pi\frac{n}{T_0}t}dt\right]^*$$

$$= x_n^*$$

- This means that for real $x(t)$, the positive and negative coefficients are conjugates. Hence, $|x_n|$ has even symmetry and $\angle x_n$ has odd symmetry with respect to the $n=0$ axis

Fourier Series and Its Properties – Real Signals (13/17)

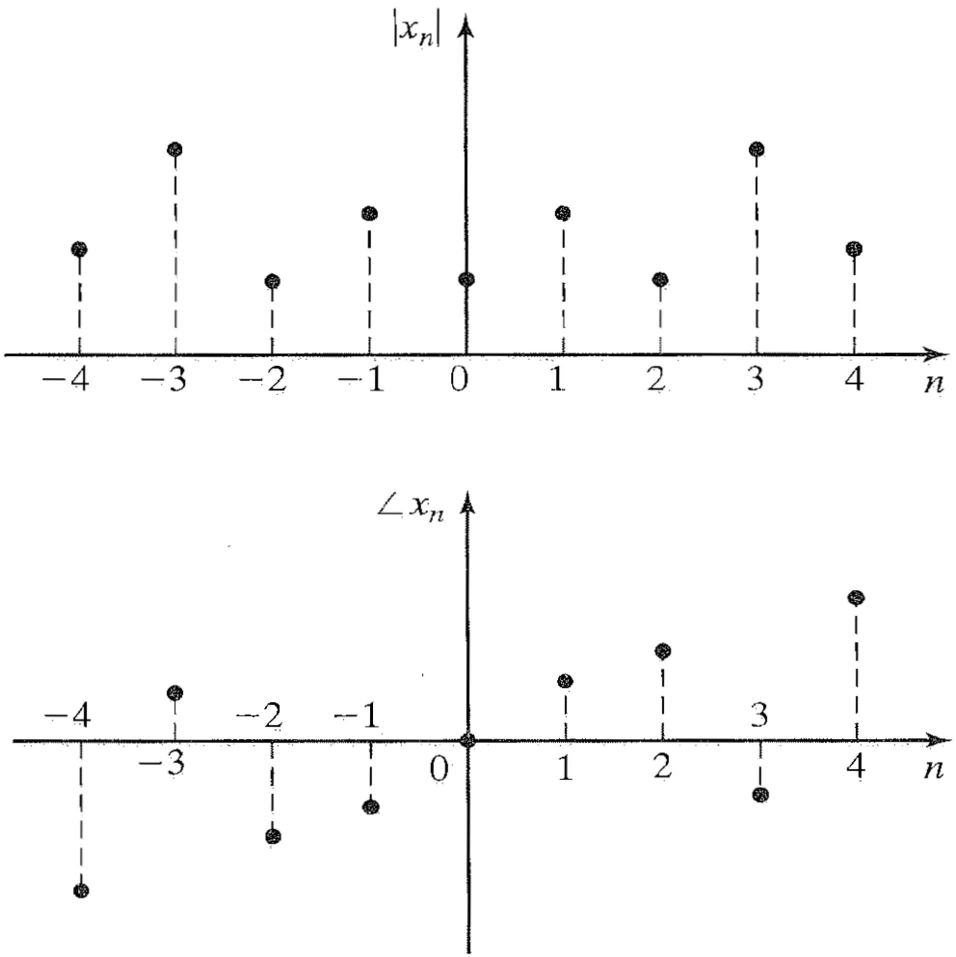

Figure 2.30 Discrete spectrum of a real-valued signal.

Fourier Series and Its Properties – Real Signals (14/17)

- In summary, for real periodic signal $x(t)$, we have three alternatives to represent the Fourier-series expansion

$$
\begin{aligned}
x(t) &= \sum_{n=-\infty}^{\infty} x_n e^{j2\pi \frac{n}{T_0} t} \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(2\pi \frac{n}{T_0} t \right) + b_n \sin\left(2\pi \frac{n}{T_0} t \right) \right] \\
&= x_0 + 2\sum_{n=1}^{\infty} \left[|x_n| \cos\left(2\pi \frac{n}{T_0} t + \angle x_n \right) \right]
\end{aligned}
$$

Fourier Series and Its Properties – Fourier-Series Expansion for Even and Odd Signals (15/17)

- If $x(t)$ is both real and even, its Fourier-series expansion has only cosine terms, i.e., we have

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(2\pi \frac{n}{T_0} t\right)$$

- On the other hand, if $x(t)$ is both real and odd, its Fourier-series expansion is

$$x(t) = \sum_{n=1}^{\infty} b_n \sin\left(2\pi \frac{n}{T_0} t\right)$$

Fourier Series and Its Properties – Response of LTI Systems to Periodic Signals (16/17)

- Assume that $x(t)$, the input to the LTI system, is periodic with period T_0 and has a Fourier-series representation

$$x(t) = \sum_{n=-\infty}^{\infty} x_n e^{j2\pi \frac{n}{T_0} t}$$

Then we have

$$\begin{aligned} y(t) &= \mathscr{L}[x(t)] \\ &= \mathscr{L}\Big[\sum_{n=-\infty}^{\infty} x_n e^{j2\pi \frac{n}{T_0} t}\Big] \\ &= \sum_{n=-\infty}^{\infty} x_n \, \mathscr{L}\big[e^{j2\pi \frac{n}{T_0} t}\big] \\ &= \sum_{n=-\infty}^{\infty} x_n H\big(\tfrac{n}{T_0}\big) e^{j2\pi \frac{n}{T_0} t}, \quad where\ H(f) = \int_{-\infty}^{\infty} h(t) e^{-j2\pi f t} dt \end{aligned}$$

Fourier Series and Its Properties – Response of LTI Systems to Periodic Signals (17/17)

- If the input to an LTI system is periodic with period T_0, then the output is also periodic. (Hint: Sum of periodic signals is a periodic signal)
- Only the frequency components that are presented at the input can be presented at the output. This means that *an LTI system cannot introduce new frequency components in the output, if these components are different from those already present at the input*
- The amount of change in amplitude $|H(n/T_0)|$ and phase $\angle H(n/T_0)$ are functions of n