Chapter 2 Signals and Linear Systems (I)

Chapter Outline

- Basic Concepts
- Fourier Series
- Fourier Transform
- Filter Design
- Power and Energy
- Hilbert Transform and Its Properties
- Lowpass and Bandpass Signals
- Further reading

Basic Operations on Signals (1/4)

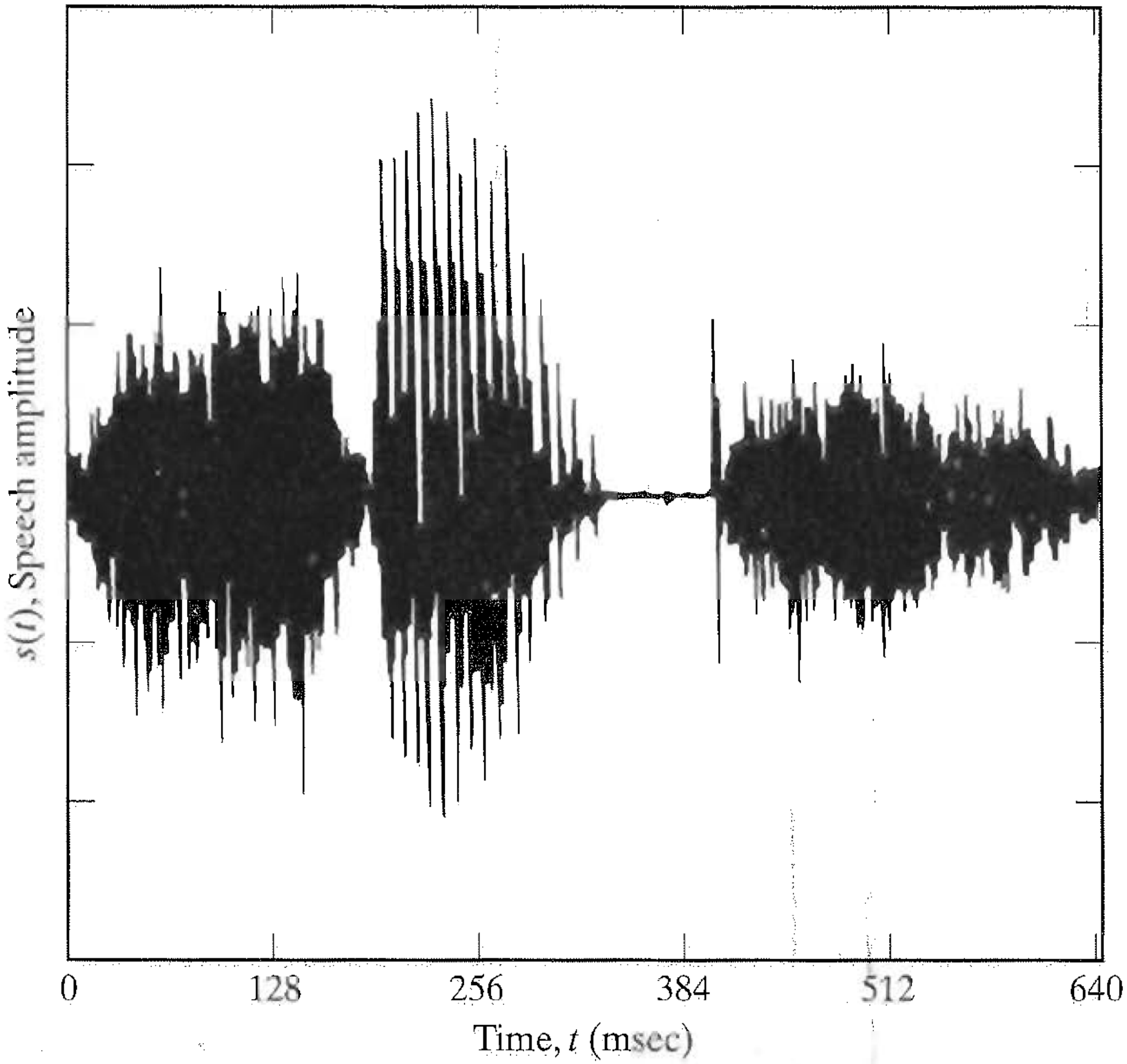

Figure 2.1 A sample speech waveform.

Basic Operations on Signals (2/4)

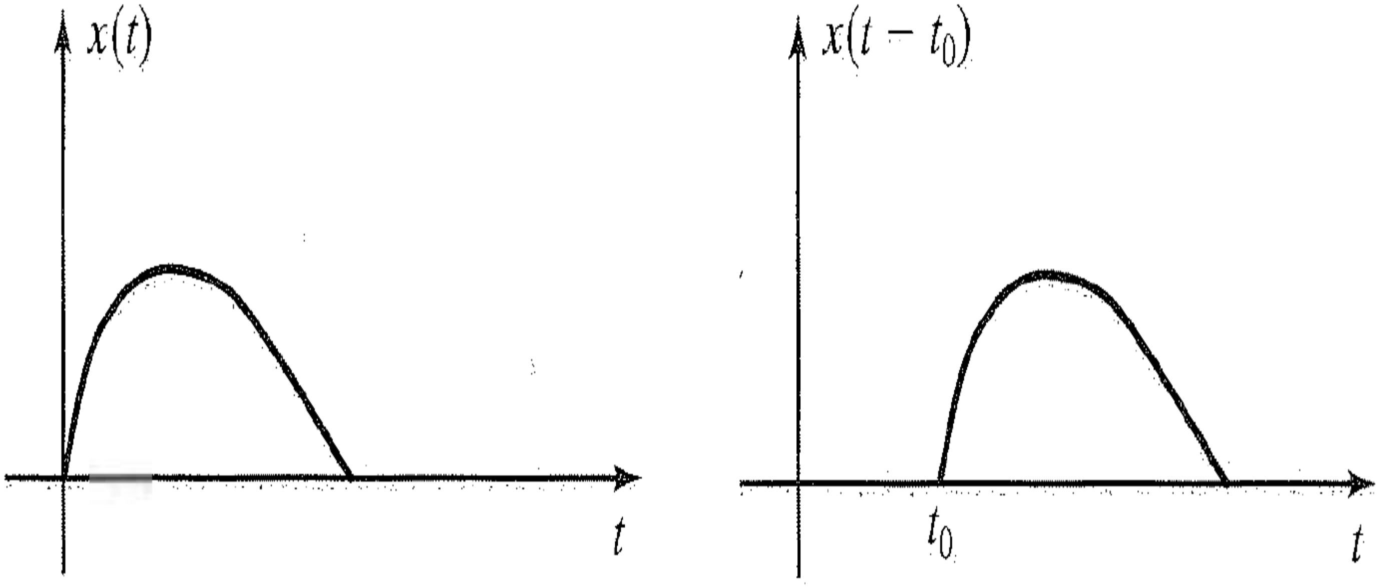

Figure 2.2 Time shifting of a signal.

Basic Operations on Signals (3/4)

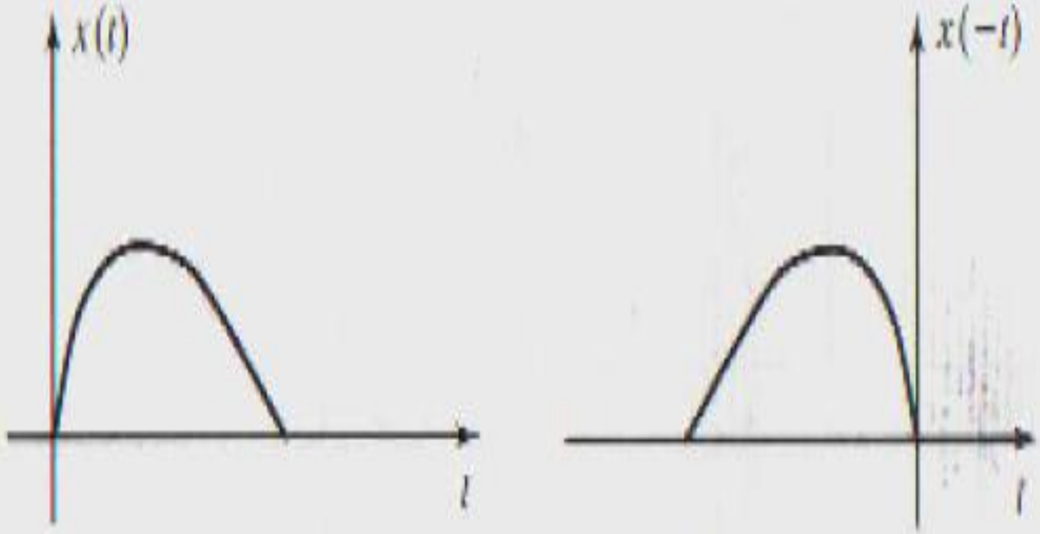

Figure 2.3 Time reversal of a signal.

Basic Operations on Signals (4/4)

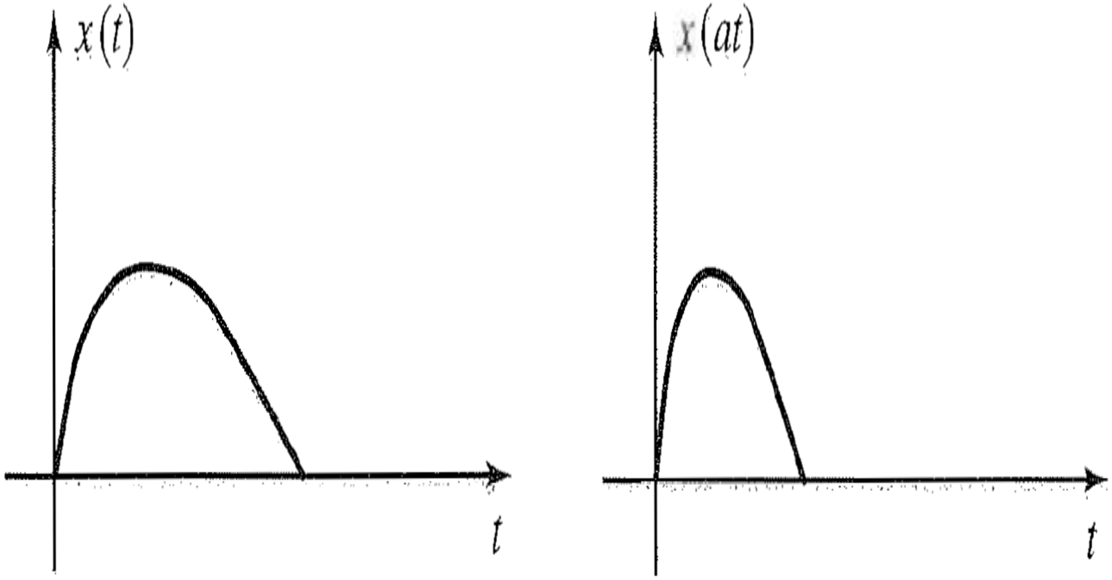

Figure 2.4 Time scaling of a signal.

Continuous-Time and Discrete-Time Signals (1/3)

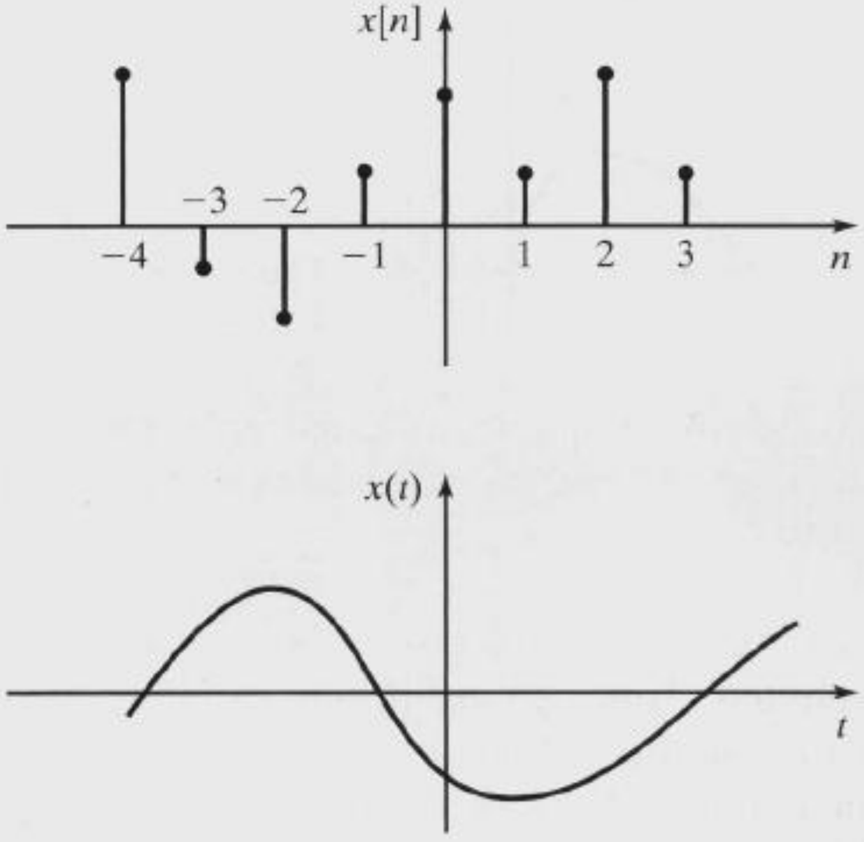

Figure 2.5 Examples of discrete-time and continuous-time signals.

Continuous-Time and Discrete-Time Signals (2/3)

- **Example 2.1.1** $x(t)=A\cos(2\pi f_0 t+\theta)$ is an example of a continuous-time signal called a *sinusoidal* signal.

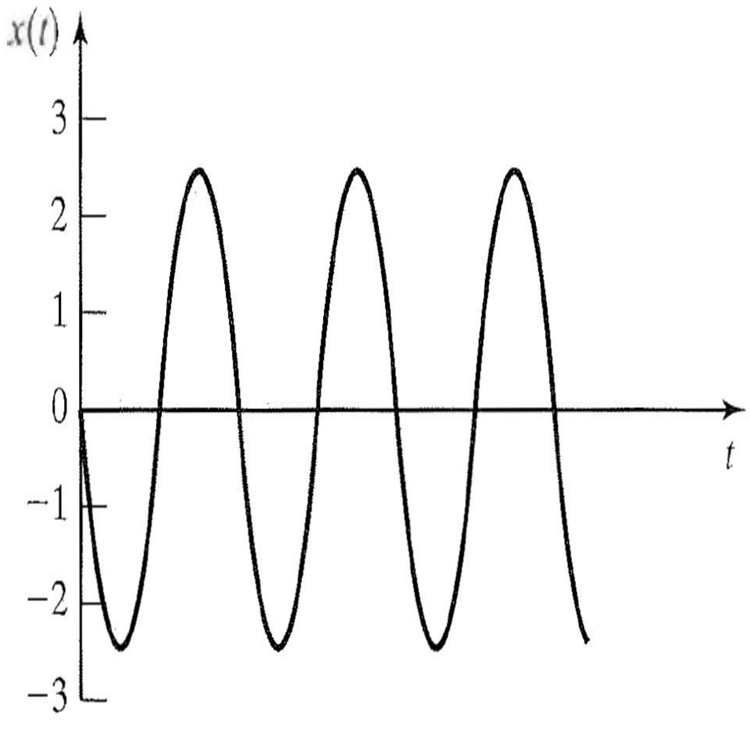

Figure 2.6 Sinusoidal signal.

Continuous-Time and Discrete-Time Signals (3/3)

- **Example 2.1.2** $x[n]=A\cos(2\pi f_0 n+\theta)$, where n belongs to the set of integers.

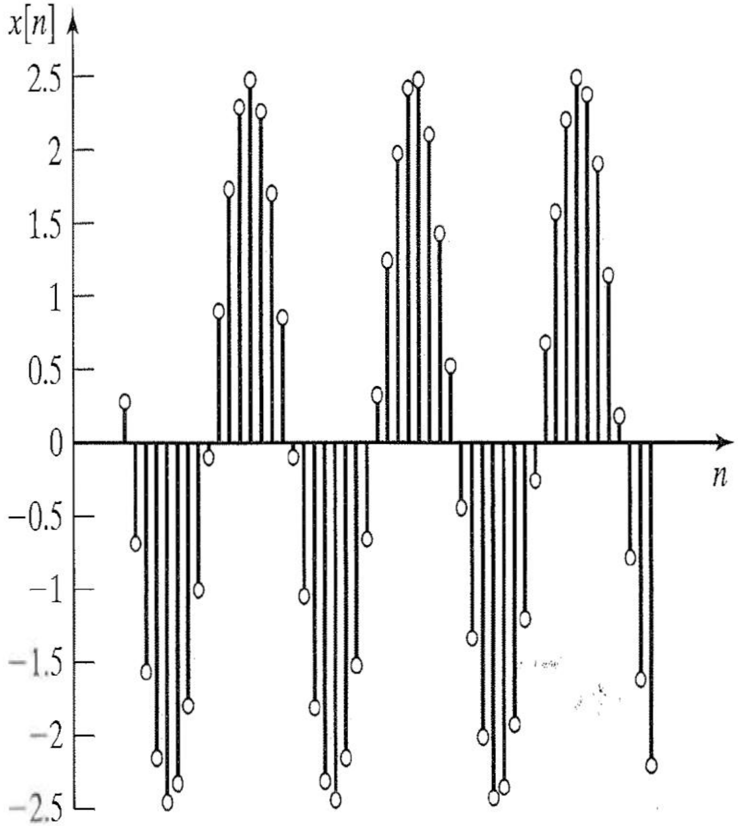

Figure 2.7 Discrete-time sinusoidal signal.

Real and Complex Signals (1/3)

- A real signal takes its values in the set of real numbers, i.e., $x(t) \in \mathbf{R}$
- A complex signal takes its values in the set of complex number, i.e., $x(t) \in \mathbf{C}$
- A complex signal can be represented by two real signals. These two real signals can be either the real and imaginary parts or the absolute value (or modulus or magnitude) and phase

Real and Complex Signals (2/3)

- **Example 2.1.3** The signal $x(t) = Ae^{j(2\pi f_0 t + \theta)}$, is a complex signal. Its real part is

$$x_r(t) = A\cos(2\pi f_0 t + \theta)$$

and its imaginary part is

$$x_i(t) = A\sin(2\pi f_0 t + \theta).$$

The absolute value of $x(t)$ is

$$|x(t)| = \sqrt{x_r^2(t) + x_i^2(t)} = |A|$$

and its phase is

$$\angle x(t) = 2\pi f_0 t + \theta.$$

Real and Complex Signals (3/3)

- **Example 2.1.3 (Cont'd)**

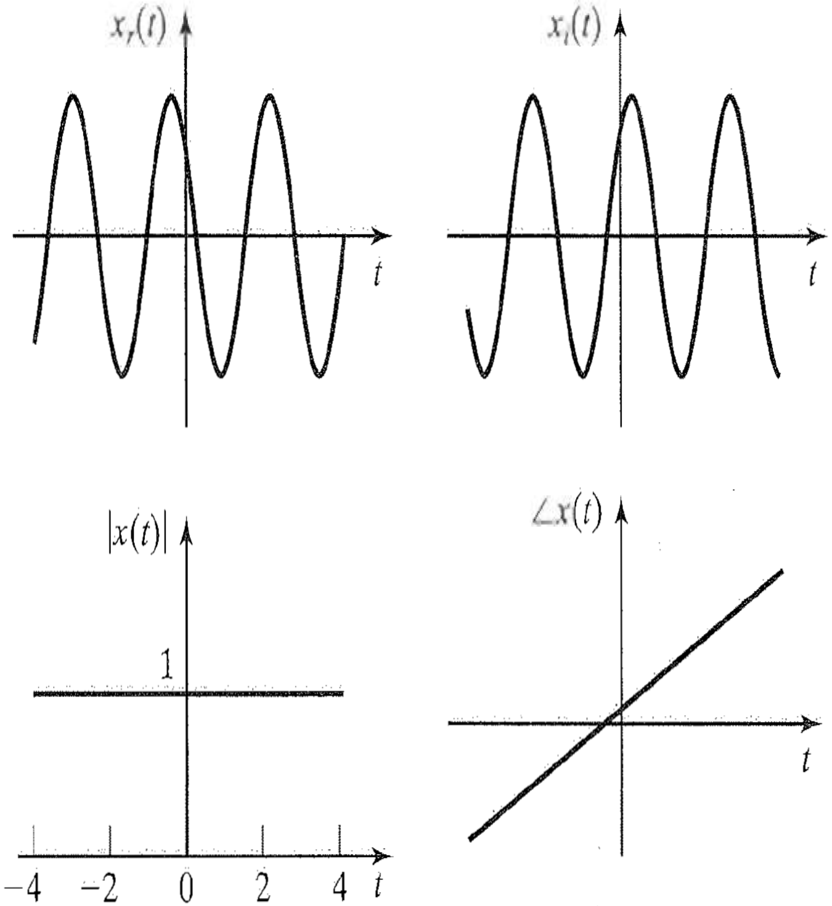

Figure 2.8 Real–imaginary and magnitude–phase graphs of the complex exponential signal in Example 2.1.3.

Deterministic and Random Signals (1/1)

- In a deterministic signal at any time instant t, the value of $x(t)$ is given as a real or a complex number
- In a random (or stochastic) signal at any given time instant t, $x(t)$ is a random variable. It is defined by a probability density function

Periodic and Nonperiodic Signals (1/3)

- A periodic signal repeats in time
- The minimum repeating interval is called *period*
- A periodic signal is a signal $x(t)$ that satisfies the property

$$x(t+T_0)=x(t)$$

for all t, and some positive real number T_0 (called the period of the signal)

- For a discrete-time period signal, we have

$$x[n+T_0]=x[n]$$

for all integers n, and a positive integer T_0 (called the period)

- A signal that does not satisfy the conditions of periodicity is called nonperiodic

Periodic and Nonperiodic Signals (2/3)

- The signal $x(t)=A\cos(2\pi f_0 t+\theta)$ and $x(t)=Ae^{j(2\pi f_0 t+\theta)}$ are periodic signals with identical period $T_0=1/f_0$
- The unit-step signal

$$u_{-1}(t)=\begin{cases}1, & t\geq 0\\ 0, & t<0\end{cases}$$

is a nonperiodic signal

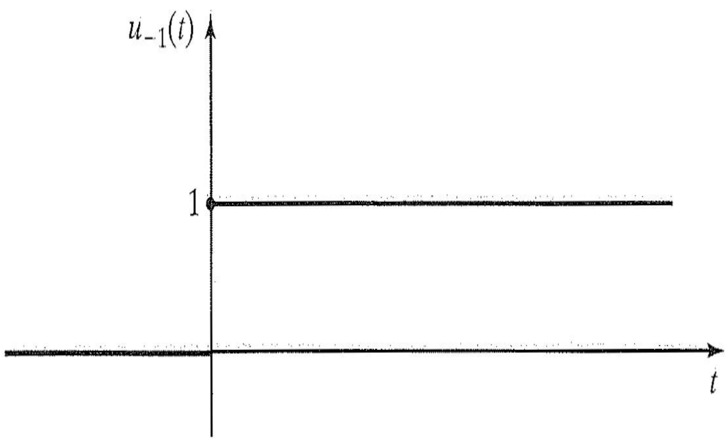

Figure 2.9 The unit-step signal.

Periodic and Nonperiodic Signals (3/3)

- **Example 2.1.5** The signal $x[n]=A\cos(2\pi f_0 n+\theta)$ is not periodic for all values of f_0. For this signal to be periodic, we must have

$$2\pi f_0(n+N_0)+\theta=2\pi f_0 n+\theta+2m\pi$$

for all integers n, some positive integer N_0, and some integer m. Thus, we conclude that

$$2\pi f_0 N_0=2m\pi$$

or

$$f_0=m/N_0$$

- The discrete sinusoidal signal is periodic only for rational values of f_0

Causal and Noncausal Signals (1/2)

- A signal $x(t)$ is called *causal* if for all $t<0$, we have $x(t)=0$; otherwise, the signal is noncausal
- A discrete-time signal is a causal signal if it is identically equal to zero for $n<0$

Causal and Noncausal Signals (2/2)

- **Example 2.1.6** The signal

$$x(t) = \begin{cases} A\cos(2\pi f_0 t + \theta), & t \geq 0 \\ 0, & otherwise \end{cases}$$

is a causal signal

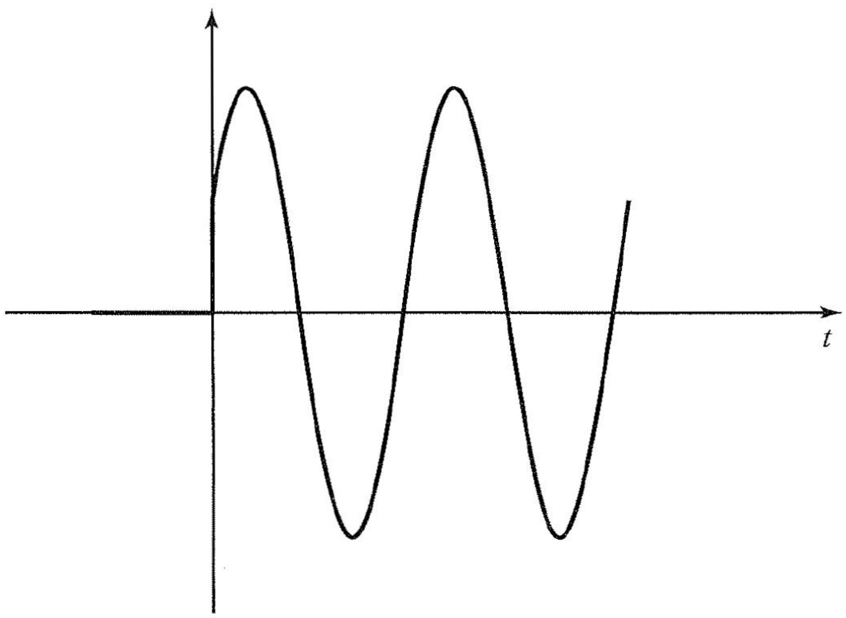

Figure 2.10 An example of a causal signal.

Even and Odd Signals (1/3)

- A signal $x(t)$ is *even* if it has mirror symmetry with respect to the vertical axis. A signal is *odd* if it is symmetric with respect to the origin
- The signal $x(t)$ is even if and only if, for all t,

$$x(-t)=x(t),$$

and is odd if and only if, for all t,

$$x(-t)=-x(t)$$

Even and Odd Signals (2/3)

- In general, any signal $x(t)$ can be written as the sum of its even and odd parts as

$$x(t) = x_e(t) + x_o(t),$$

where

$$x_e(t) = \frac{x(t) + x(-t)}{2},$$

$$x_o(t) = \frac{x(t) - x(-t)}{2}$$

Even and Odd Signals (3/3)

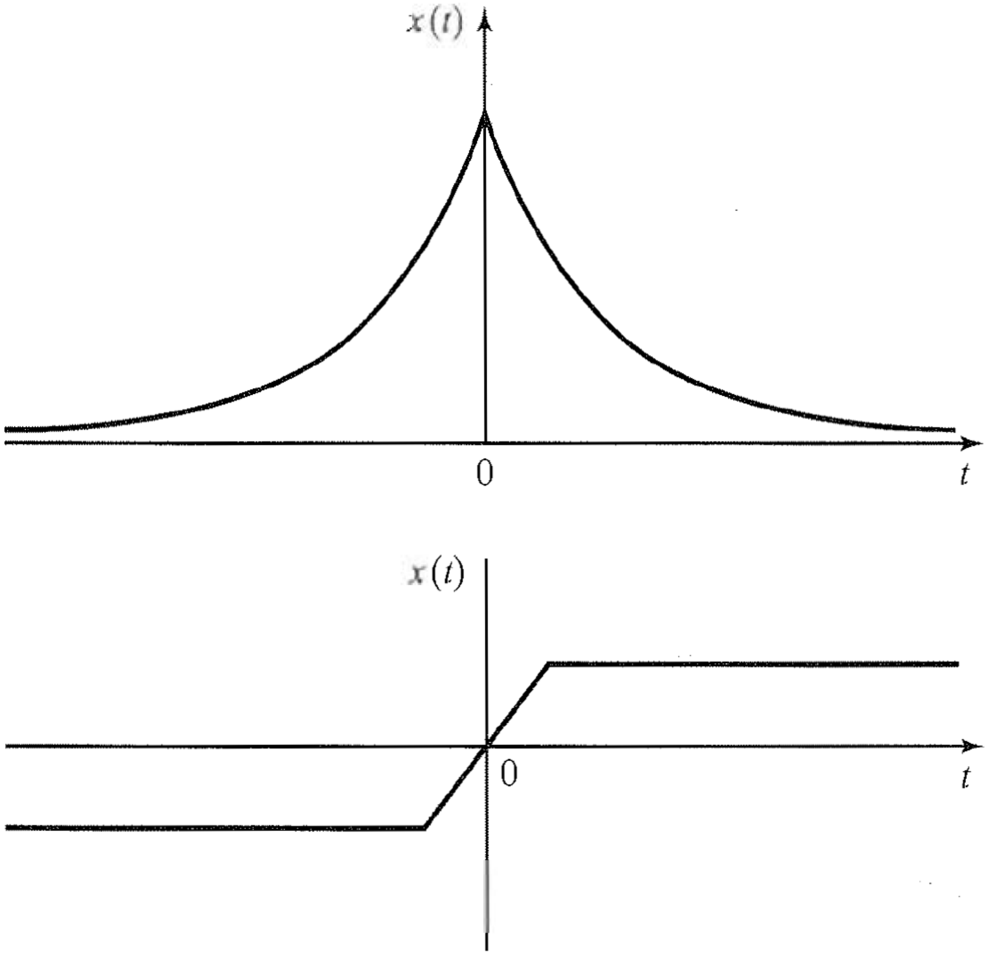

Figure 2.11 Examples of even and odd signals.

Hermitian Symmetry for Complex Signals (1/1)

- A complex signal $x(t)$ is called Hermitian if its real part is even and its imaginary part is odd
- We can easily show that its magnitude is even and its phase is odd.
- The signal $x(t)=Ae^{j2\pi f_0 t}$ is an example of a Hermitian signal

Energy-Type and Power-Type Signals (1/5)

- This classification deals with the energy content and the power content of signals. Before classifying these signals, we need to define the energy content (or simply the energy) and the power content (or power)
- The energy content of the signal is defined by

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \lim_{T \to \infty} \int_{-T/2}^{T/2} |x(t)|^2 \, dt$$

- The power content is defined by

$$P_x = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt$$

- For real signal, $|x(t)|^2$ is replaced by $x^2(t)$

Energy-Type and Power-Type Signals (2/5)

- A signal $x(t)$ is an energy-type signal if and only if E_x is finite
- A signal $x(t)$ is a power-type signal if and only if $0<P_x<\infty$
- **Example 2.1.9** Find the energy in the signal described by

$$x(t)=\begin{cases}3, & |t|<3\\ 0, & otherwise.\end{cases}$$

We have

$$E_x=\int_{-\infty}^{\infty}|x(t)|^2\,dt=\int_{-3}^{3}9dt=54.$$

This signal is an energy-type signal

Energy-Type and Power-Type Signals (3/5)

- **Example 2.1.10** The energy content of $A\cos(2\pi f_0 t+\theta)$ is

$$E_x = \lim_{T\to\infty}\int_{-T/2}^{T/2} A^2\cos^2(2\pi f_0 t+\theta)\,dt = \infty$$

This signal is not an energy-type signal. The power of this signal is

$$P_x = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} A^2\cos^2(2\pi f_0 t+\theta)\,dt$$

$$= \frac{A^2}{2} < \infty$$

Hence, $x(t)$ is a power-type signal and its power is $\frac{A^2}{2}$

Energy-Type and Power-Type Signals (4/5)

- **Example 2.1.11** For any periodic signal with period T_0, the energy is

$$\begin{aligned} E_x &= \lim_{T\to\infty} \int_{-T/2}^{T/2} |x(t)|^2\, dt \\ &= \lim_{n\to\infty} \int_{-nT_0/2}^{nT_0/2} |x(t)|^2\, dt \\ &= \lim_{n\to\infty} n \int_{-T_0/2}^{T_0/2} |x(t)|^2\, dt \\ &= \infty \end{aligned}$$

Therefore, periodic signals are not energy-type signals

Energy-Type and Power-Type Signals (5/5)

- **Example 2.1.11 (Cont'd)** The power content of any periodic signal is

$$
\begin{aligned}
P_x &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt \\
&= \lim_{n\to\infty} \frac{1}{nT_0} \int_{-nT_0/2}^{nT_0/2} |x(t)|^2 \, dt \\
&= \lim_{n\to\infty} \frac{n}{nT_0} \int_{-T_0/2}^{T_0/2} |x(t)|^2 \, dt \\
&= \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} |x(t)|^2 \, dt
\end{aligned}
$$

The power content of a periodic signal is equal to the average power in one period

Some Important Signals and Their Properties (1/18)

- **The Sinusoidal Signal.** The sinusoidal signal is defined by

$$x(t) = A\cos(2\pi f_0 t + \theta)$$

where the parameters A, f_0, and θ are the amplitude, frequency, and phase of the signal

- The period is $T_0 = 1/f_0$

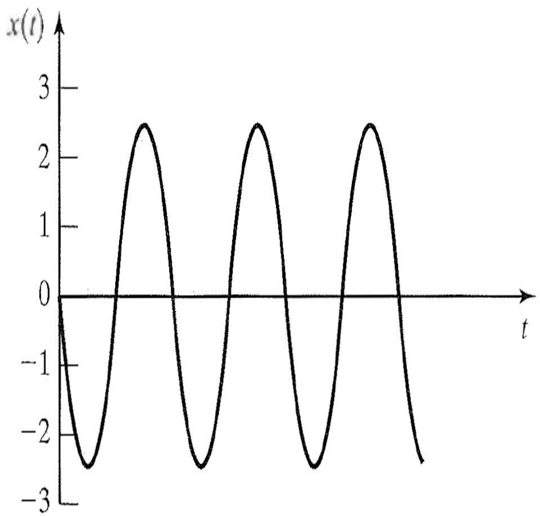

Figure 2.6 Sinusoidal signal.

Some Important Signals and Their Properties (2/18)

- **The complex exponential signal.** The complex exponential signal is defined by $x(t)=Ae^{j(2\pi f_0 t+\theta)}$
- A, f_0, and θ are the amplitude, frequency, and phase of the signal

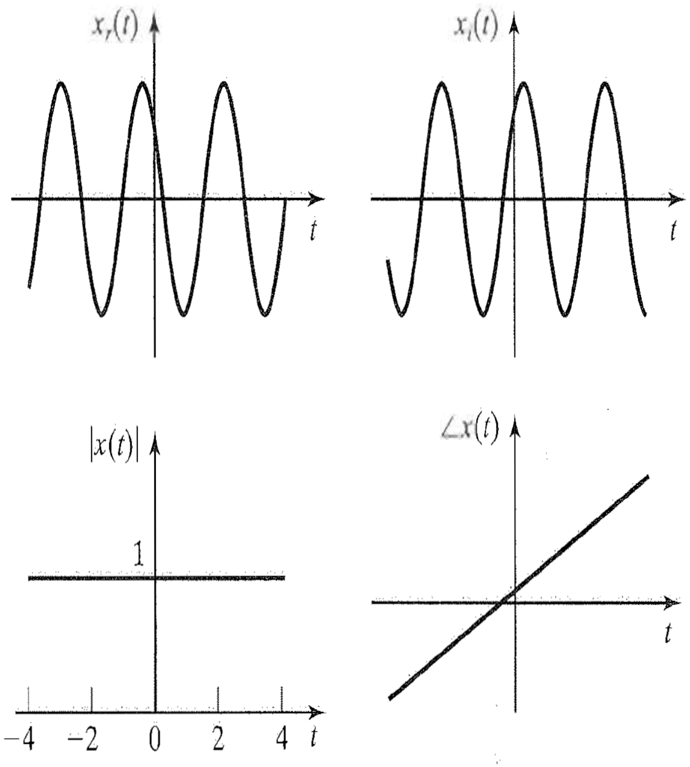

Figure 2.8 Real–imaginary and magnitude–phase graphs of the complex exponential signal in Example 2.1.3.

Some Important Signals and Their Properties (3/18)

- **The Unit-Step Signal.** The unit step multiplied by any signal produces a "causal version" of the signal
- For positive a, we have $u_{-1}(at)=u_{-1}(t)$

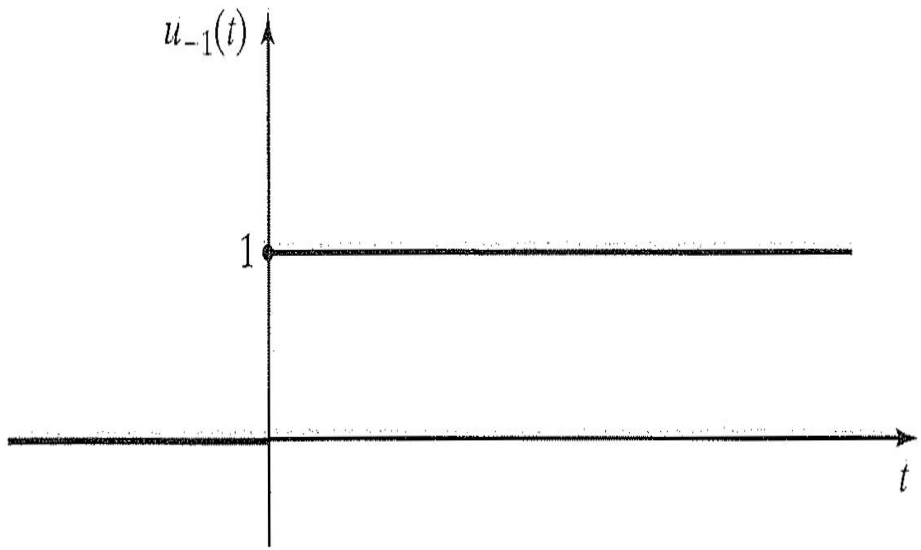

Figure 2.9 The unit-step signal.

Some Important Signals and Their Properties (4/18)

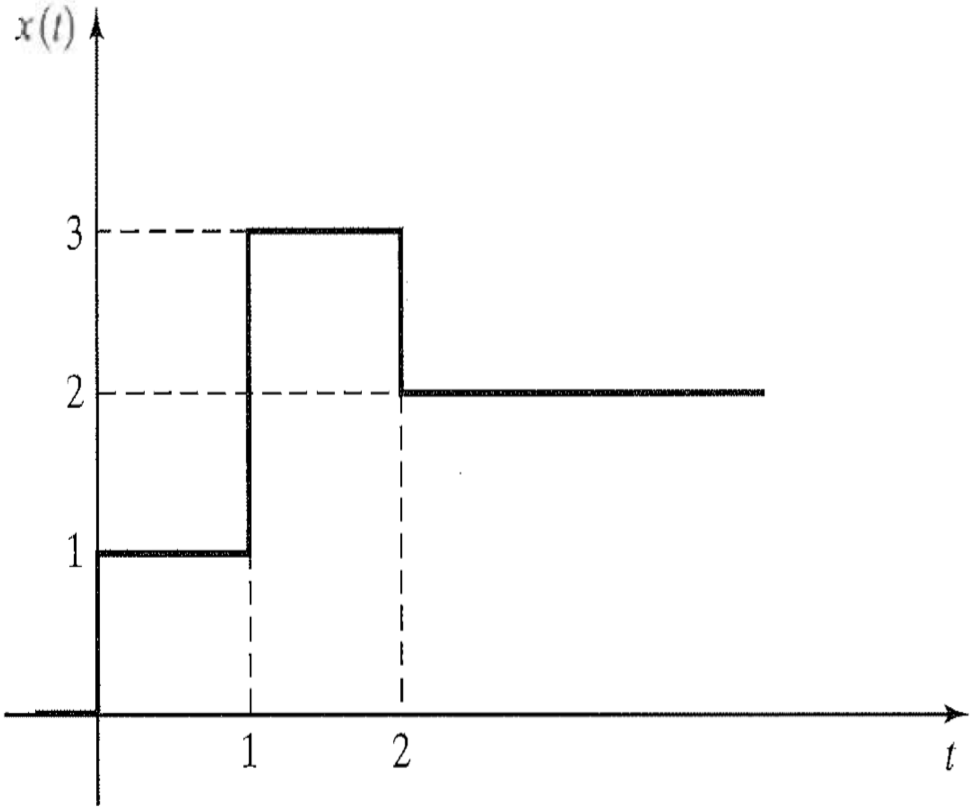

Figure 2.12 The signal $u_{-1}(t) + 2u_{-1}(t-1) - u_{-1}(t-2)$.

Some Important Signals and Their Properties (5/18)

- **The Rectangular Pulse.** This signal is defined as

$$\Pi(t) = \begin{cases} 1, & -\frac{1}{2} \le t \le \frac{1}{2} \\ 0, & otherwise \end{cases}$$

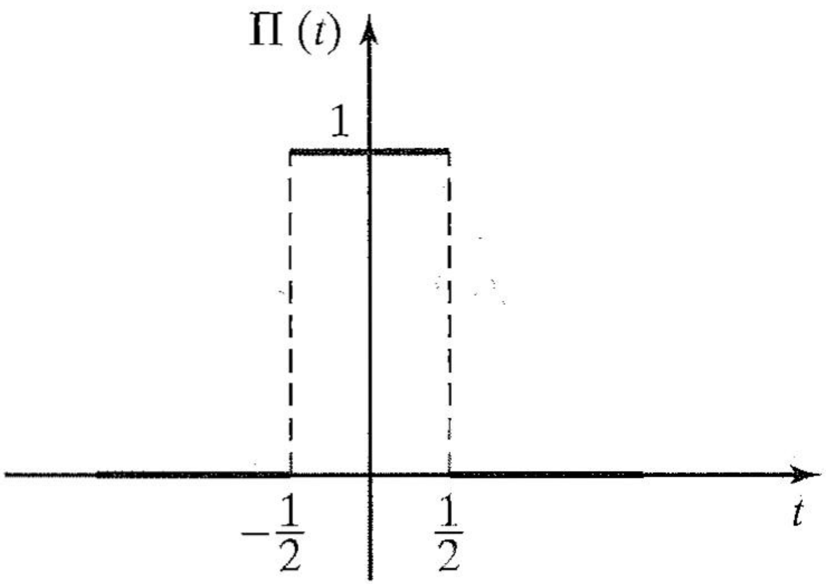

Figure 2.13 The rectangular pulse.

Some Important Signals and Their Properties (6/18)

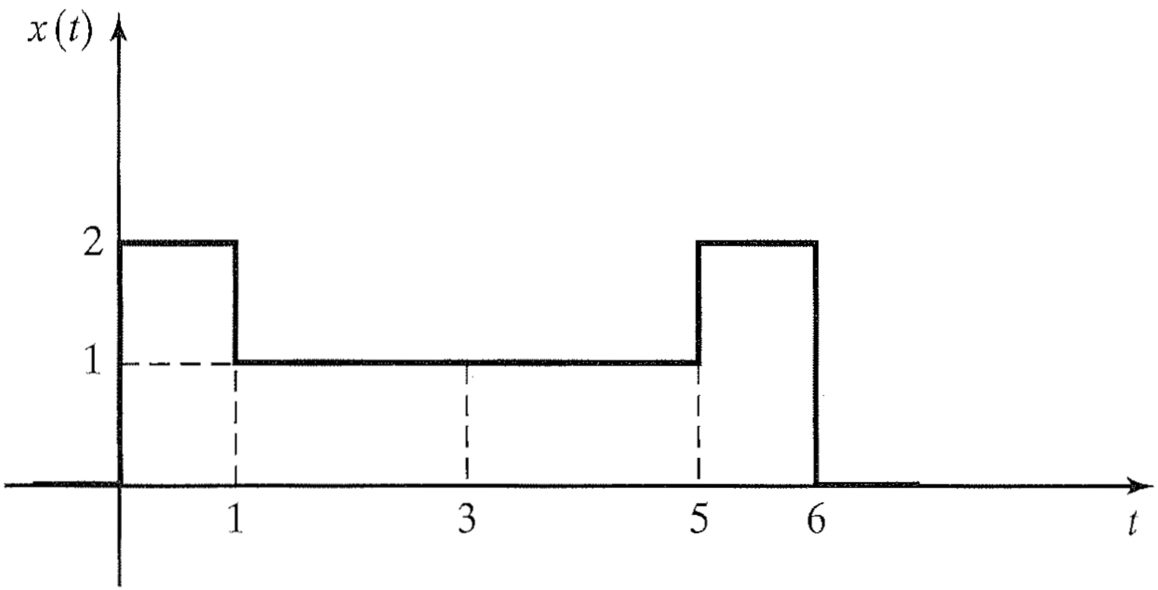

Figure 2.14 The signal $2\Pi\left(\frac{t-3}{6}\right) - \Pi\left(\frac{t-3}{4}\right)$.

Some Important Signals and Their Properties (7/18)

- **The Triangular Signal.** The signal is defined as

$$\Lambda(t) = \begin{cases} t+1, & -1 \le t \le 0 \\ -t+1, & 0 \le t \le 1 \\ 0, & otherwise \end{cases}$$

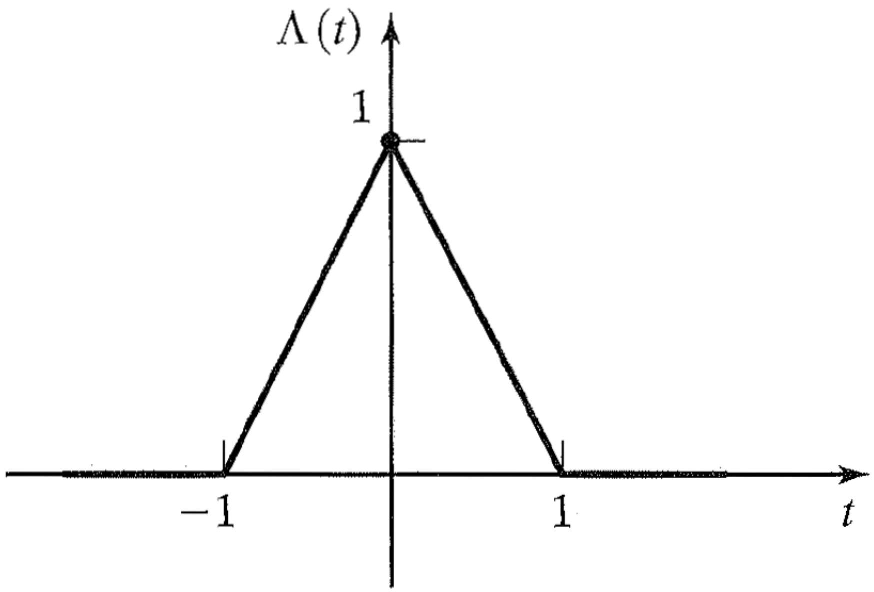

Figure 2.15 The triangular signal.

Some Important Signals and Their Properties (8/18)

- **Example 2.1.14.** Plot $\Pi(\frac{t}{4}) + \Lambda(\frac{t}{2})$

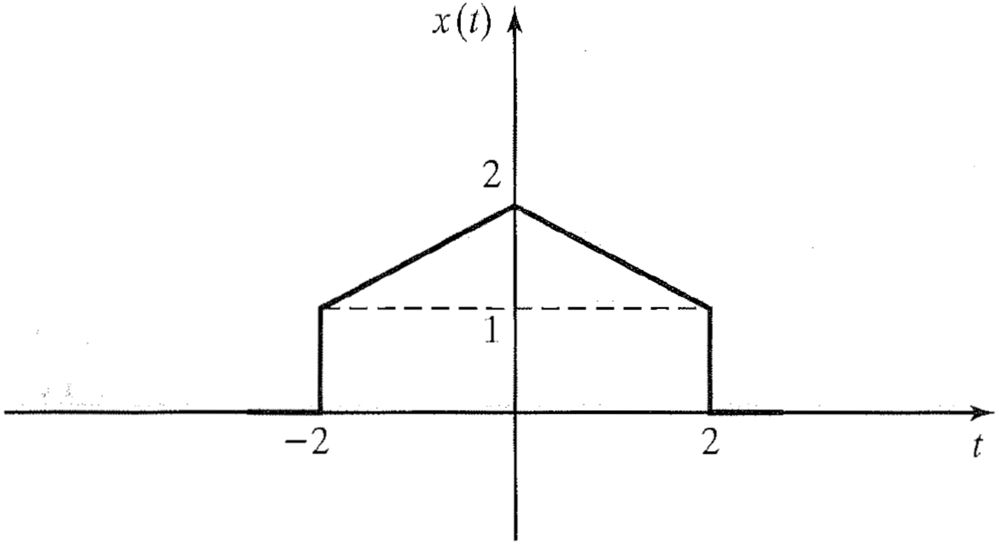

Figure 2.16 The signal $\Pi\left(\frac{t}{4}\right) + \Lambda\left(\frac{t}{2}\right)$.

Its plot is shown in Figure 2.15. It is not difficult to verify that[2]

$$\Lambda(t) = \Pi(t) \star \Pi(t). \tag{2.1.17}$$

Some Important Signals and Their Properties (9/18)

- **The Sinc Signals.** The sinc signal is defined as

$$\text{sinc}(t) = \begin{cases} \dfrac{\sin(\pi t)}{\pi t}, & t \neq 0 \\ 1, & t = 0 \end{cases}$$

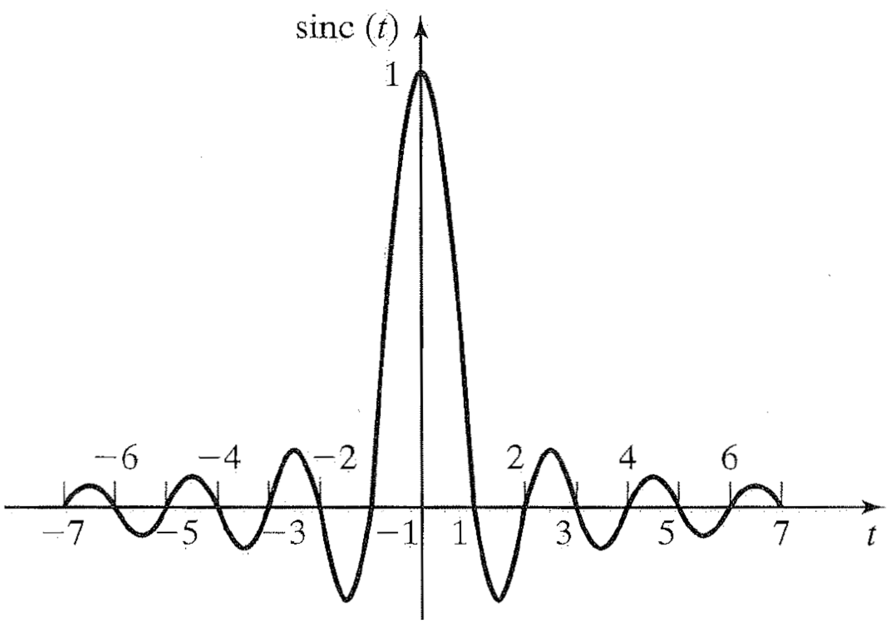

Figure 2.17 The sinc signal.

Some Important Signals and Their Properties (10/18)

- **The Sign or the Signum Signal.** The sign or the signum signal is defined as

$$\operatorname{sgn}(t) = \begin{cases} 1, & t > 0 \\ -1, & t < 0 \\ 0, & t = 0 \end{cases}$$

- The signum signal can be expressed as the limit of the signal $x_n(t)$, which is defined by

$$x_n(t) = \begin{cases} e^{-\frac{t}{n}}, & t > 0 \\ -e^{\frac{t}{n}}, & t < 0 \\ 0, & t = 0 \end{cases}$$

when $n \to \infty$

Some Important Signals and Their Properties (11/18)

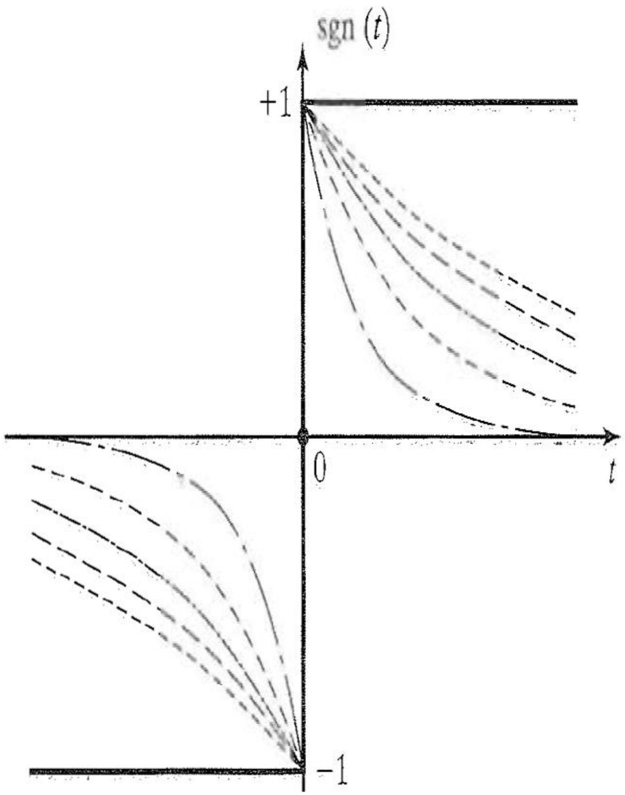

Figure 2.18 The signum signal as the limit of $x_n(t)$.

Some Important Signals and Their Properties — Properties of $\delta(t)$ (12/18)

- **The Impulse or Delta Signal.** The impulse signal is not a function (or signal). It is a *distribution* or a *generalized function*.
- A distribution is defined in terms of its effect on another function under the integral sign
- The impulse distribution can be defined by the relation

$$\int_{-\infty}^{\infty} \phi(t)\delta(t)dt = \phi(0)$$

- Sometimes it is helpful to visualize $\delta(t)$ as the limit of certain known signals such as

$$\delta(t) = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \Pi\left(\frac{t}{\varepsilon}\right)$$

and

$$\delta(t) = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \operatorname{sinc}\left(\frac{t}{\varepsilon}\right)$$

Some Important Signals and Their Properties — Properties of $\delta(t)$ (13/18)

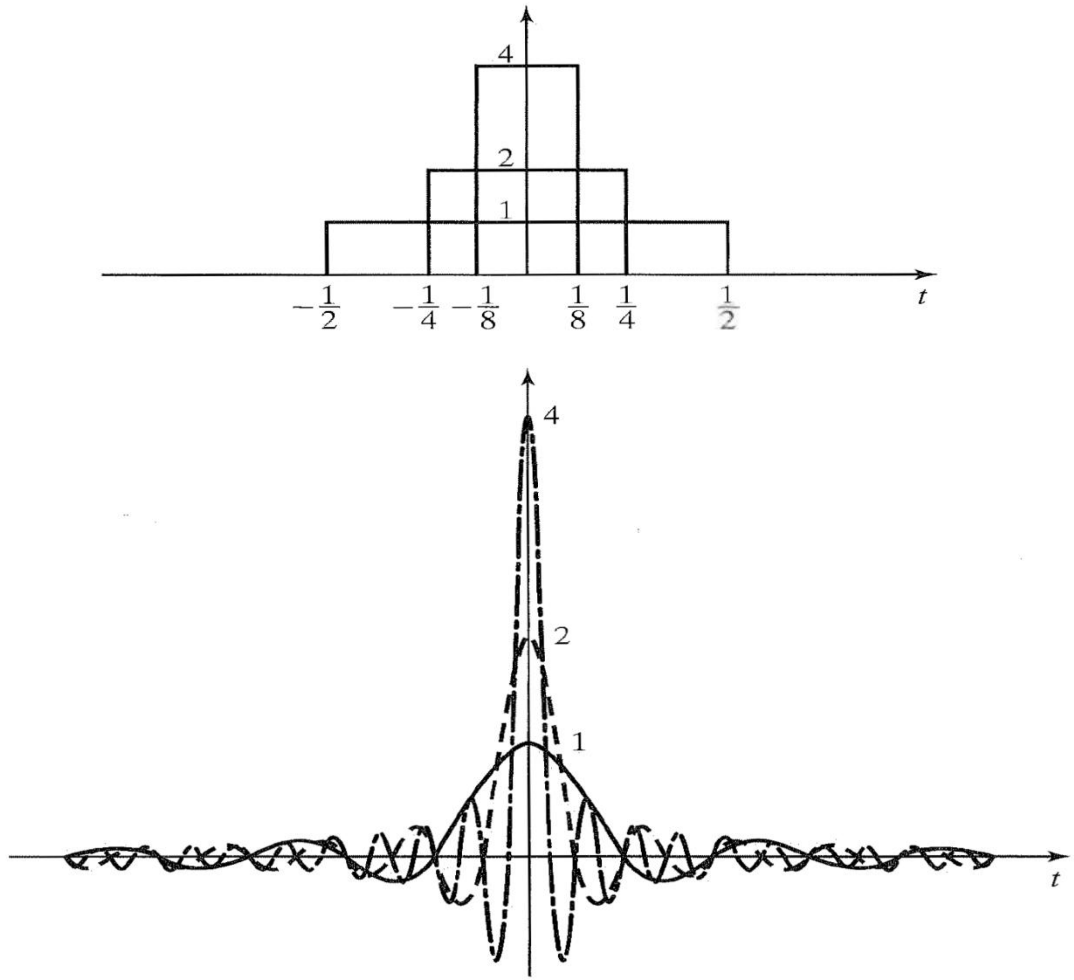

Figure 2.19 The impulse signal as a limit.

Some Important Signals and Their Properties — Properties of $\delta(t)$ (14/18)

- $\delta(t)=0$ for all $t \neq 0$ and $\delta(0)=\infty$
- $x(t)\delta(t-t_0)=x(t_0)\delta(t-t_0)$
- For any $\phi(t)$ continuous at t_0,

$$\int_{-\infty}^{\infty} \phi(t)\delta(t-t_0)dt = \phi(t_0)$$

- For any $\phi(t)$ continuous at t_0,

$$\int_{-\infty}^{\infty} \phi(t+t_0)\delta(t)dt = \phi(t_0)$$

- For all $a \neq 0$,

$$\delta(at) = \frac{1}{|a|}\delta(t)$$

Some Important Signals and Their Properties — Properties of $\delta(t)$ (15/18)

- The result of the convolution of any signal with the impulse signal is the signal itself

$$x(t) \bigstar \delta(t) = x(t)$$

$$x(t) \bigstar \delta(t-t_0) = x(t-t_0)$$

- The unit-step signal is the integral of the impulse signal. The impulse signal is the *generalized derivatives* of the unit-step signal

$$u_{-1}(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$

$$\delta(t) = \frac{d}{dt} u_{-1}(t)$$

Some Important Signals and Their Properties — Properties of $\delta(t)$ (16/18)

- We define the generalized derivatives of $\delta(t)$ by

$$\int_{-\infty}^{\infty} \delta^{(n)}(t)\phi(t)dt = (-1)^n \frac{d^n}{dt^n}\phi(t)\bigg|_{t=0}$$

We can generalize this result to

$$\int_{-\infty}^{\infty} \delta^{(n)}(t-t_0)\phi(t)dt = (-1)^n \frac{d^n}{dt^n}\phi(t)\bigg|_{t=t_0}$$

[Hint of proof]

$$\int_{-\infty}^{\infty} \delta^{(1)}(t-t_0)\phi(t)dt$$

$$= \int_{-\infty}^{\infty} \phi(t)d\delta(t-t_0)$$

$$= \phi(t)\delta(t-t_0)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta(t-t_0)d\phi(t)$$

$$= -\phi^{(1)}(t_0)$$

Some Important Signals and Their Properties — Properties of $\delta(t)$ (17/18)

- The result of the convolution of any signal with nth derivative of $x(t)$ is the nth derivative of $x(t)$

$$x(t) \bigstar \delta^{(n)}(t) = x^{(n)}(t)$$

[Hint of proof] By mathematical induction,

$$\begin{aligned}
& x(t) \bigstar \delta^{(1)}(t) \\
&= \int_{-\infty}^{\infty} x(\tau)\delta^{(1)}(t-\tau)d\tau \\
&= -\int_{-\infty}^{\infty} x(\tau)d\delta(t-\tau) \\
&= -x(\tau)\delta(t-\tau)\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \delta(t-\tau)dx(\tau) \\
&= x^{(1)}(t)
\end{aligned}$$

Some Important Signals and Their Properties — Properties of $\delta(t)$ (18/18)

- The result of the convolution of any signal $x(t)$ with the unit-step signal is the integral of the signal $x(t)$

$$x(t) \bigstar u_{-1}(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

- For even values of n, $\delta^{(n)}(t)$ is even; for odd values of n, it is odd

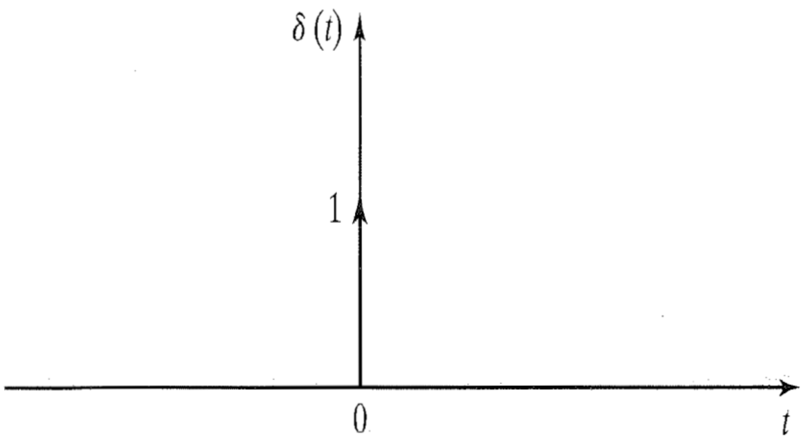

Figure 2.20 The impulse signal.