SIGNALS AND SYSTEMS

PROBLEMS SET 3

[Please do your best to work out ALL but the first 3 problems, and you are encouraged to discuss your solutions with your classmates.]

1. Consider a continuous-time LTI system with impulse response $h(t)=e^{-4|t|}$. Find the Fourier series representation of the output $y(t)$ for each of the following inputs: (a) $x(t)=\sum_{n=-\infty}^{\infty}\delta(t-n)$; (b) $x(t)=\sum_{n=-\infty}^{\infty}(-1)^n\delta(t-n)$; (c) $x(t)$ is the periodic signal depicted in Fig. P1.

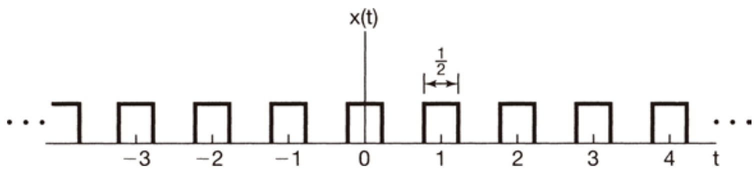

Fig. P1

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The frequency response of the system is given by

$$H(j\omega)=\int_{-\infty}^{\infty}e^{-4|t|}e^{-j\omega t}dt=\frac{1}{4+j\omega}+\frac{1}{4-j\omega}.$$

(a) Here, $T=1$ and $\omega_0=2\pi$ and $a_k=1$ for all k. The FS coefficients of the output are

$$b_k=a_kH(jk\omega_0)=\frac{1}{4+j2\pi k}+\frac{1}{4-j2\pi k}.$$

(b) Here, $T=2$ and $\omega_0=\pi$ and

$$a_k=\begin{cases}0, & k \text{ even}\\ 1, & k \text{ odd}\end{cases}.$$

Therefore, the FS coefficients of the output are

$$b_k=a_kH(jk\omega_0)=\begin{cases}0, & k \text{ even}\\ \frac{1}{4+j\pi k}+\frac{1}{4-j\pi k}, & k \text{ odd}\end{cases}.$$

(c) Here, $T = 1$, $\omega_0 = 2\pi$ and

$$a_k = \begin{cases} 1/2, & k = 0 \\ 0, & k \text{ even}, k \neq 0 \\ \frac{\sin(\pi k/2)}{\pi k}, & k \text{ odd} \end{cases}.$$

Therefore, the FS coefficients of the output are

$$b_k = a_k H(jk\omega_0) = \begin{cases} 1/4, & k = 0 \\ 0, & k \text{ even}, k \neq 0 \\ \frac{\sin(\pi k/2)}{\pi k}\left[\frac{1}{4+j2\pi k} + \frac{1}{4-j2\pi k}\right], & k \text{ odd} \end{cases}.$$

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2. Consider a continuous time LTI system whose frequency response is $H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt = \dfrac{\sin(4\omega)}{\omega}$. If the input to this system is a periodic signal $x(t) = \begin{cases} 1, & 0 \le t < 4 \\ -1, & 4 \le t < 8 \end{cases}$ with period $T = 8$, determine the corresponding system output $y(t)$.

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Let us first evaluate the Fourier series coefficients of $x(t)$. Clearly, since $x(t)$ is real and odd, a_k is purely imaginary and odd. Therefore, $a_0 = 0$. Now,

$$\begin{aligned} a_k &= \frac{1}{8}\int_0^8 x(t)e^{-j(2\pi/8)kt}dt \\ &= \frac{1}{8}\int_0^4 e^{-j(2\pi/8)kt}dt - \frac{1}{8}\int_4^8 e^{-j(2\pi/8)kt}dt \\ &= \frac{1}{j\pi k}[1 - e^{-j\pi k}] \end{aligned}$$

Clearly, the above expression evaluates to zero for all even values of k. Therefore,

$$a_k = \begin{cases} 0, & k = 0, \pm 2, \pm 4, \cdots \\ \frac{2}{j\pi k}, & k = \pm 1, \pm 3, \pm 5, \cdots \end{cases}$$

When $x(t)$ is passed through an LTI system with frequency response $H(j\omega)$, the output $y(t)$ is given by (see Section 3.8)

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0)e^{jk\omega_0 t}$$

where $\omega_0 = \frac{2\pi}{T} = \frac{\pi}{4}$. Since a_k is non zero only for odd values of k, we need to evaluate the above summation only for odd k. Furthermore, note that

$$H(jk\omega_0) = H(jk(\pi/4)) = \frac{\sin(k\pi)}{k(\pi/4)}$$

is always zero for odd values of k. Therefore,

$$y(t) = 0.$$

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3. A causal and stable LTI system S has the frequency response $H(j\omega)=\dfrac{j\omega+4}{6-\omega^2+5j\omega}$ (a) Determine a differential equation relating the input $x(t)$ and output $y(t)$ of S; (b) Determine the impulse response $h(t)$ of S; (iii) What is the output of S when the input is $x(t)=e^{-4t}u(t)-te^{-4t}u(t)$.

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(a) We have

$$\frac{Y(j\omega)}{X(j\omega)}=\frac{j\omega+4}{6-\omega^2+5j\omega}.$$

Cross-multiplying and taking the inverse Fourier transform, we obtain

$$\frac{d^2y(t)}{dt^2}+5\frac{dy(t)}{dt}+6y(t)=\frac{dx(t)}{dt}+4x(t).$$

(b) We have

$$H(j\omega)=\frac{2}{2+j\omega}-\frac{1}{3+j\omega}.$$

Taking the inverse Fourier transform we obtain,

$$h(t)=2e^{-2t}u(t)-e^{-3t}u(t).$$

(c) We have

$$X(j\omega)=\frac{1}{4+j\omega}-\frac{1}{(4+j\omega)^2}.$$

Therefore,

$$Y(j\omega)=X(j\omega)H(j\omega)=\frac{1}{(4+j\omega)(2+j\omega)}.$$

Finding the partial fraction expansion of $Y(j\omega)$ and taking the inverse Fourier transform,

$$y(t)=\frac{1}{2}e^{-2t}u(t)-\frac{1}{2}e^{-4t}u(t).$$

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4. Suppose we are given the following information about a continuous-time signal $x(t)$ with period T=6 and Fourier coefficients a_k : (i) $x(t)$ is a real signal; (ii) $a_k=0$ for $k=0$ and $k>2$; (iii) $x(t)=-x(t-3)$; (iv) $\frac{1}{6}\int_{-3}^{3}|x(t)|^2\,dt=0.5$; (v) a_1 is a positive real number. Show that $x(t)=A\cos(Bt+C)$ and determine the values of the constants A, B, and C.

[$x(t)=cos(\pi t/3)$

The only unknown FS coefficients are a_1, a_{-1}, a_2, and a_{-2}. Since $x(t)$ is real, $a_1 = a^*_{-1}$ and $a_2 = a^*_{-2}$. Since a_1 is real, $a_1 = a_{-1}$. Now, $x(t)$ is of the form

$$x(t) = A_1 \cos(\omega_0 t) + A_2 \cos(2\omega_0 t + \theta),$$

where $\omega_0 = 2\pi/6$. From this we get

$$x(t-3) = A_1 \cos(\omega_0 t - 3\omega_0) + A_2 \cos(2\omega_0 t + \theta - 6\omega_0).$$

Now if we need $x(t) = -x(t-3)$, then $3\omega_0$ and $6\omega_0$ should both be odd multiples of π. Clearly, this is impossible. Therefore, $a_2 = a_{-2} = 0$ and

$$x(t) = A_1 \cos(\omega_0 t).$$

Now, using Parseval's relation on Clue 5, we get

$$\sum_{k=-\infty}^{\infty} |a_k|^2 = |a_1|^2 + |a_{-1}|^2 = \frac{1}{2}.$$

Therefore, $|a_1| = 1/2$. Since a_1 is positive, we have $a_1 = a_{-1} = 1/2$. Therefore, $x(t) = \cos(\pi t/3)$.

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5. Let $x(t)$ be a signal with Fourier transform $X(j\omega)$. Suppose we are given the following facts: (i) $x(t)$ is real; (ii) $x(t) = 0 \ for \ t \le 0$; (iii) $\frac{1}{2\pi}\int_{-\infty}^{\infty} \Re e\{X(j\omega)\} e^{j\omega t} d\omega = |t| e^{-|t|}$. Determine a closed-form expression for $x(t)$.

[$x(t) = 2te^{-t}u(t)$

Since $x(t)$ is real,

$$\mathcal{E}v\{x(t)\} = \frac{x(t)+x(-t)}{2} \stackrel{FT}{\longleftrightarrow} \mathcal{R}e\{X(j\omega)\}.$$

We are given that

$$\mathcal{IFT}\{\mathcal{R}e\{X(j\omega)\}\} = |t|e^{-|t|}.$$

Therefore,

$$\mathcal{E}v\{x(t)\} = \frac{x(t)+x(-t)}{2} = |t|e^{-|t|}.$$

We also know that $x(t) = 0$ for $t \le 0$. This implies that $x(-t)$ is zero for $t > 0$. We may conclude that

$$x(t) = 2|t|e^{-|t|} \qquad \text{for } t \ge 0$$

Therefore,

$$x(t) = 2te^{-t}u(t)$$

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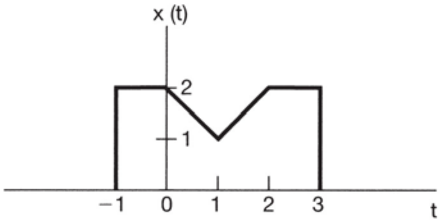
6. Consider a causal LTI system with frequency response $H(j\omega) = \frac{1}{3+j\omega}$. For a particular input $x(t)$, this system id observed to produce the output $y(t) = e^{-3t}u(t) - e^{-4t}u(t)$. Determine $x(t)$. [$x(t) = e^{-4t}u(t)$]

7. Let $X(j\omega)$ denote the Fourier transform of the signal $x(t)$ depicted in Fig. P3.

You should perform all these calculations without explicitly evaluating $X(j\omega)$.



(a) $X(j\omega)$ can be expressed as $A(j\omega)e^{j\theta(j\omega)}$, where $A(j\omega)$ and $\theta(j\omega)$ are both real-values. Find $\theta(j\omega)$. [$\theta(t) = -\omega$]

(b) Find $X(j0)$. [$X(j0) = 7$]

(c) Find $\int_{-\infty}^{\infty} X(j\omega)d\omega$. [4π]

(d) Evaluate $\int_{-\infty}^{\infty} X(j\omega)\frac{2\sin\omega}{\omega}e^{j2\omega}d\omega$. [7π]

(e) Evaluate $\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$. [26π]

(f) Sketch the inverse Fourier transform of $\Re e\{X(j\omega)\}$.

[

(a) Note that $y(t) = x(t+1)$ is a real and even signal. Therefore, $Y(j\omega)$ is also real and even. This implies that $\sphericalangle Y(j\omega) = 0$. Also, since $Y(j\omega) = e^{j\omega}X(j\omega)$, we know that $\sphericalangle X(j\omega) = -\omega$.

(b) We have

$$X(j0) = \int_{-\infty}^{\infty} x(t)dt = 7.$$

(c) We have

$$\int_{-\infty}^{\infty} X(j\omega)d\omega = 2\pi x(0) = 4\pi.$$

(d) Let $Y(j\omega) = \frac{2\sin\omega}{\omega}e^{2j\omega}$. The corresponding signal $y(t)$ is

$$y(t) = \begin{cases} 1, & -3 < t < -1 \\ 0, & \text{otherwise} \end{cases}.$$

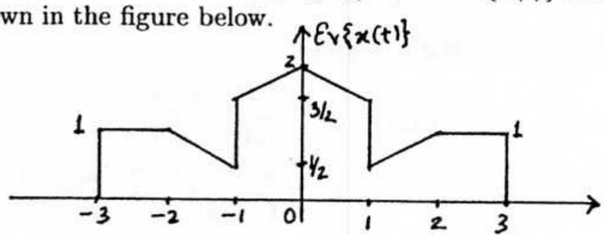
Then the given integral is

$$\int_{-\infty}^{\infty} X(j\omega)Y(j\omega)d\omega = 2\pi\{x(t) * y(t)\}_{t=0} = 7\pi.$$

(e) We have

$$\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega = 2\pi\int_{-\infty}^{\infty} |x(t)|^2 dt = 26\pi.$$

(f) The inverse Fourier transform of $\mathcal{R}e\{X(j\omega)\}$ is the $\mathcal{E}v\{x(t)\}$ which is $[x(t)+x(-t)]/2$. This is as shown in the figure below.



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8. Suppose $g(t) = x(t)\cos t$ and the Fourier transform of the $g(t)$ is

$$G(j\omega) = \begin{cases} 1, & |\omega| \le 2 \\ 0, & otherwise \end{cases}.$$

(a) Determine $x(t)$. [$x(t) = \dfrac{2\sin t}{\pi t}$]

(b) Specify the Fourier transform $X_1(j\omega)$ of a signal $x_1(t)$ such that $g(t) = x_1(t)\cos\left(\frac{2}{3}t\right)$

[

(a) We know that

$$w(t) = \cos t \overset{FT}{\longleftrightarrow} W(j\omega) = \pi[\delta(\omega - 1) + \delta(\omega + 1)]$$

and

$$g(t) = x(t)\cos t \overset{FT}{\longleftrightarrow} G(j\omega) = \frac{1}{2\pi}\{X(j\omega) * W(j\omega)\}.$$

Therefore,

$$G(j\omega) = \frac{1}{2}X(j(\omega - 1)) + \frac{1}{2}X(j(\omega + 1)).$$

Since $G(j\omega)$ is as shown in Figure S4.30, it is clear from the above equation that $X(j\omega)$ is as shown in the Figure S4.30.

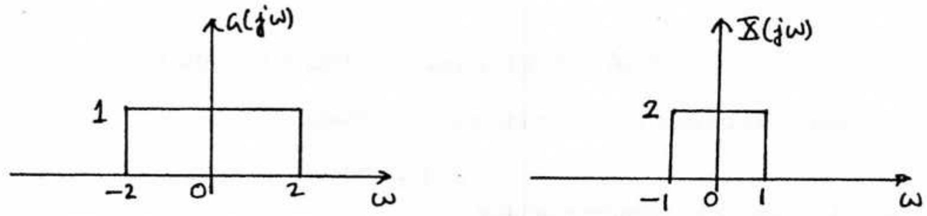

Figure S4.30

Therefore,

$$x(t) = \frac{2\sin t}{\pi t}.$$

(b) $X_1(j\omega)$ is as shown in Figure S4.30.

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9. Consider an LTI system whose response to the input $x(t) = \left[e^{-t} + 3e^{-3t}\right]u(t)$ is $y(t) = \left[2e^{-t} - 2e^{-4t}\right]u(t)$.

(a) Find the frequency response of this system. [$H(j\omega) = \dfrac{3(3 + j\omega)}{(4 + j\omega)(2 + j\omega)}$]

(b) Determine the system's impulse response. [$h(t) = \dfrac{3}{2}\left[e^{-4t} + e^{-2t}\right]u(t)$]

(c) Find the differential equation relating the input and the output of this system. [$y'' + 6y' + 8y = 3x' + 9x$]

[

4.36. **(a)** The frequency response is

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{3(3+j\omega)}{(4+j\omega)(2+j\omega)}.$$

(b) Finding the partial fraction expansion of answer in part (a) and taking its inverse Fourier transform, we obtain

$$h(t) = \frac{3}{2}\left[e^{-4t} + e^{-2t}\right]u(t).$$

(c) We have

$$\frac{Y(j\omega)}{X(j\omega)} = \frac{(9+3j\omega)}{8+6j\omega-\omega^2}.$$

Cross-multiplying and taking the inverse Fourier transform, we obtain

$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = 3\frac{dx(t)}{dt} + 9x(t).$$

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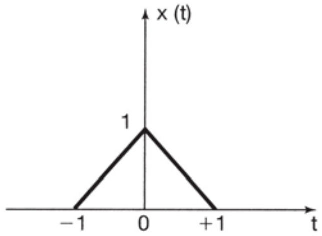
10. Consider the signal in Fig. P6.



(a) Find the Fourier transform $X(j\omega)$ of $x(t)$. [$X(j\omega) = \left[2\frac{\sin(\omega/2)}{\omega}\right]^2$]

(b) Sketch the signal $\tilde{x}(t) = x(t) * \sum_{k=-\infty}^{\infty} \delta(t-4k)$.

[

(a) Note that

$$x(t) = x_1(t) * x_1(t),$$

where

$$x_1(t) = \begin{cases} 1, & |\omega| < \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

Also, the Fourier transform $X_1(j\omega)$ of $x_1(t)$ is

$$X_1(j\omega) = 2\frac{\sin(\omega/2)}{\omega}.$$

Using the convolution property we have

$$X(j\omega) = X_1(j\omega)X_1(j\omega) = \left[2\frac{\sin(\omega/2)}{\omega}\right]^2.$$

(b) The signal $\tilde{x}(t)$ is as shown in Figure S4.37

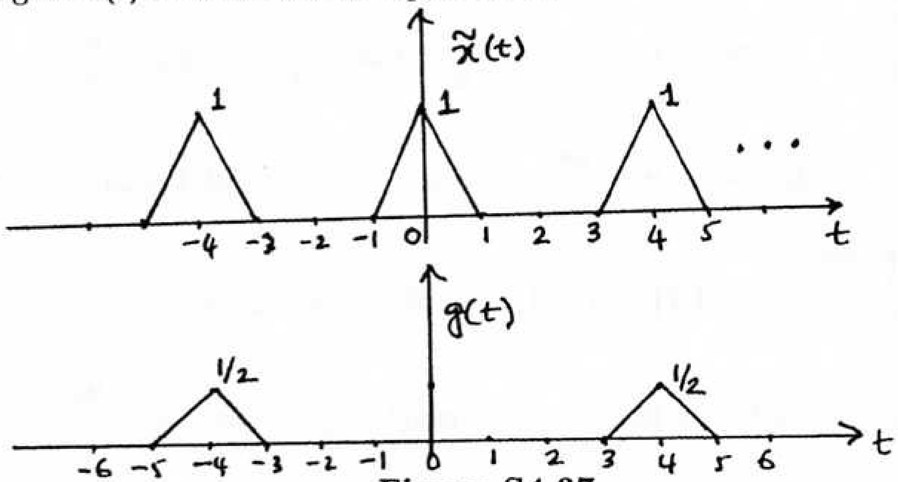


Figure S4.37

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11. Consider an LTI system with impulse response $h(t) = \pi \frac{\sin t}{\pi t} \frac{\sin 5t}{\pi t}$. Plot the frequency response of this system and determine the respective output $y_i(t)$ for each of the following inputs $x_i(t)$. Show all work. Write a closed-form expression for the output in each case.

(a) $x_1(t) = \cos(6t)$. [0]

(b) $x_2(t) = \sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k \sin(3kt)$. [$\frac{1}{2}\sin(3t)$]

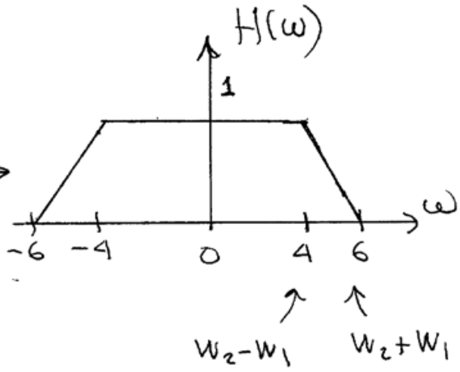
(c) $x_3(t) = \frac{\sin(4t)}{\pi t}$. [$\frac{\sin(4t)}{\pi t}$]

(d) $x_4(t) = \left[\frac{\sin(2t)}{\pi t}\right]^2$ [$\left[\frac{\sin(2t)}{\pi t}\right]^2$]

[

Prob. 2

$$h(t) = \pi \frac{\sin(t)}{\pi t} \frac{\sin(5t)}{\pi t} \overset{\mathcal{F}}{\longleftrightarrow}$$



$W_1 = 1$ $W_2 = 5$

(a)

$$x_1(t) = \cos(6t) \rightarrow \boxed{H(\omega)} \rightarrow y_1(t) = |H(6)| \cdot \cos(6t + \angle H(6))$$

$$= 0$$

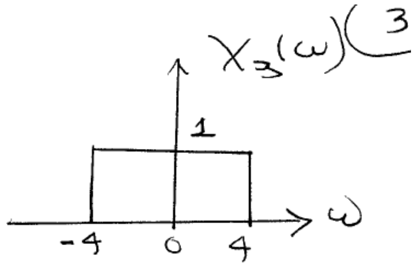
(b) $k=0$ term is zero for all t

$$x_2(t) = \left(\tfrac{1}{2}\right)\sin(3t) + \left(\tfrac{1}{4}\right)\sin(6t) + \left(\tfrac{1}{8}\right)\sin(9t) + \dots$$

only $\omega = 3$ makes it thru filter

$$y_2(t) = \tfrac{1}{2}\sin(3t)$$

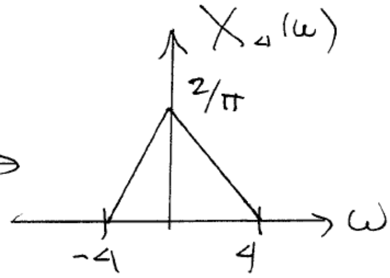
Prob. 2 Sol'n (cont.)



(c) $x_3(t) = \dfrac{\sin(4t)}{\pi t} \overset{\mathcal{F}}{\longleftrightarrow}$

THUS: $Y_3(\omega) = X_3(\omega)\, H(\omega) = X_3(\omega)$

$$y_3(t) = \frac{\sin(4t)}{\pi t}$$



(d) $x_4(t) = \left(\dfrac{\sin(2t)}{\pi t}\right)^2 \overset{\mathcal{F}}{\longleftrightarrow}$

$$Y_4(\omega) = X_4(\omega)\, H(\omega) = X_4(\omega)$$

$$y_4(t) = \left(\frac{\sin(2t)}{\pi t}\right)^2$$

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