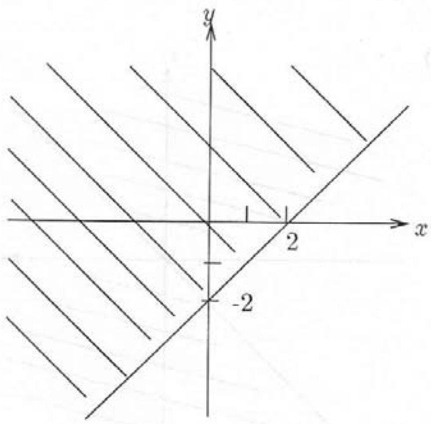
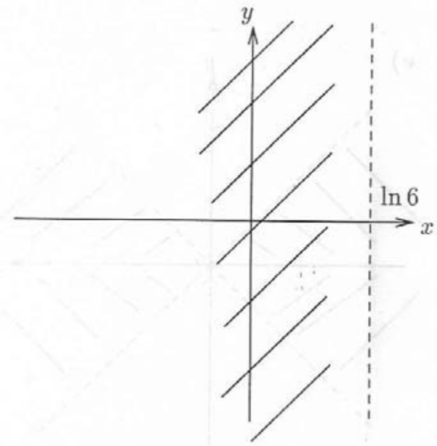
5.8 a) $\{Y \geq X - 2\}$



Not product-form

b) $\{X < \ln b\}$



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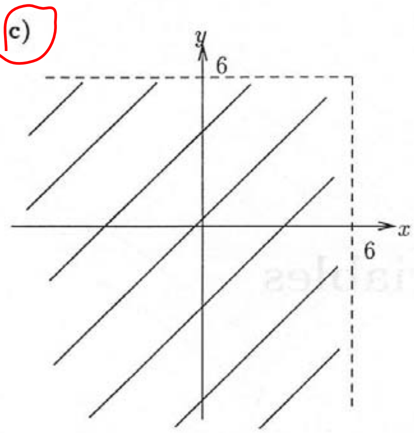
c)



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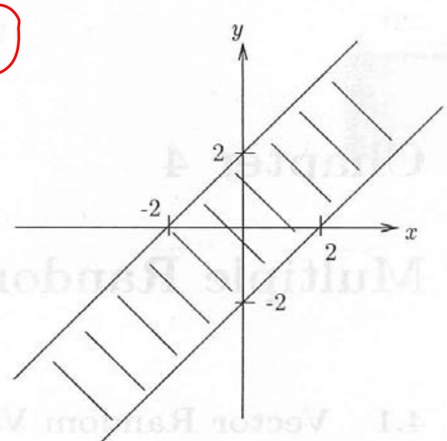
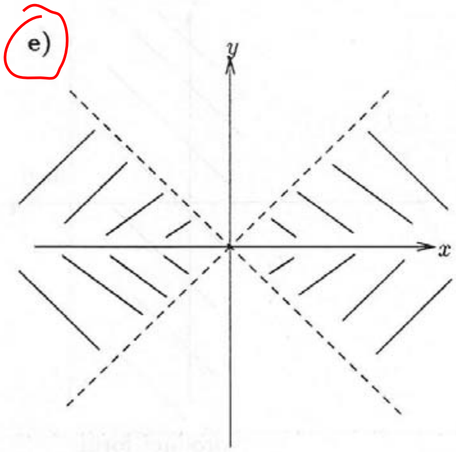
d)



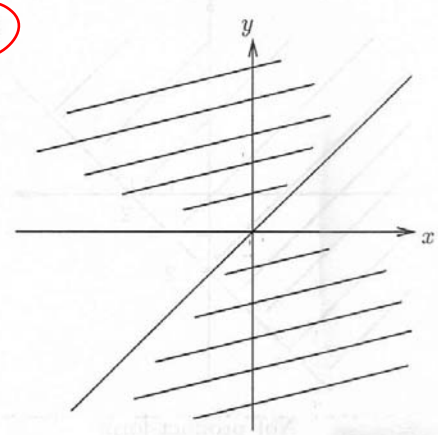
Not product-form

e)



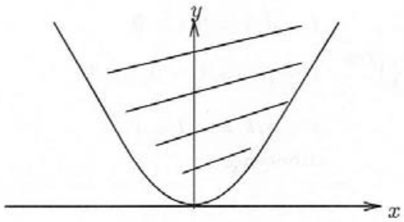
Not product-form

f)



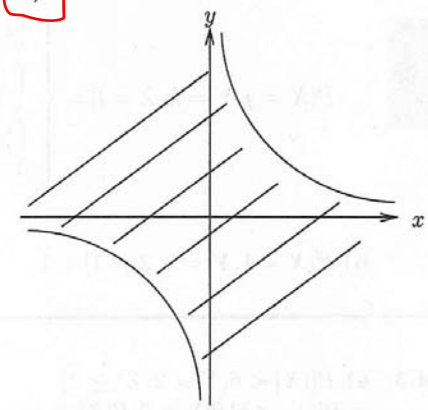
Not product-form

g)



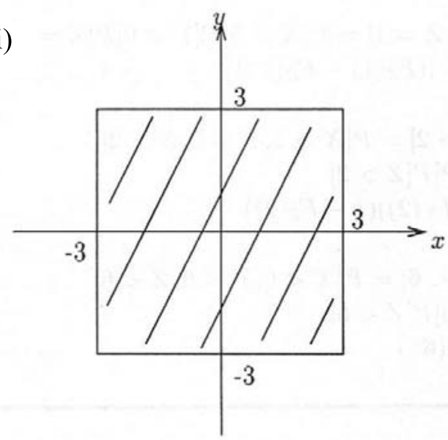
Not product-form

h)



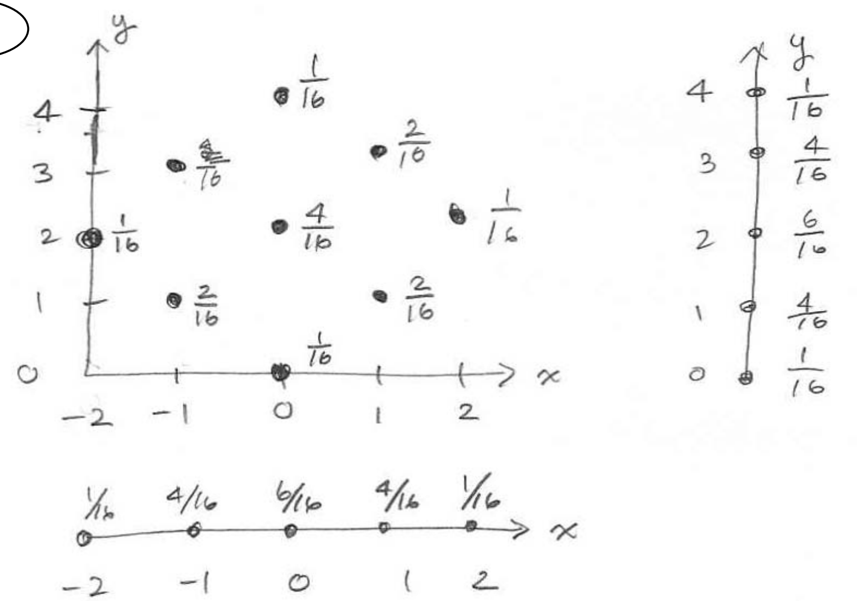
Not product-form

i)



product-form

5.11



5.12 **a)** All three cases have the same marginal pmf

$$P[X=-1] = P[X=0] = P[X=1] = \frac{1}{3}$$
$$P[Y=-1] = P[Y=0] = P[Y=1] = \frac{1}{3}$$

b)

	i	ii	iii
$P[A]$	2/3	2/3	2/3
$P[B]$	5/6	2/3	2/3
$P[C]$	2/3	1/3	1

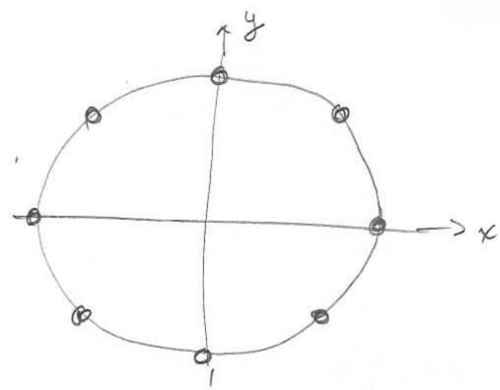
5.13

(a) Mapping S to S_{XY}

Θ	0	1	2	3	4	5	6	7
X, Y	$(r, 0)$	$(\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}})$	$(0, r)$	$(\frac{-r}{\sqrt{2}}, \frac{r}{\sqrt{2}})$	$(-r, 0)$	$(\frac{-r}{\sqrt{2}}, \frac{-r}{\sqrt{2}})$	$(0, -r)$	$(\frac{r}{\sqrt{2}}, \frac{-r}{\sqrt{2}})$

(b) Joint PMF

$X \backslash Y$	$-r$	$\frac{-r}{\sqrt{2}}$	0	$\frac{r}{\sqrt{2}}$	r
$-r$	0	0	1/8	0	0
$\frac{-r}{\sqrt{2}}$	0	1/8	0	1/8	0
0	1/8	0	0	0	1/8
$\frac{r}{\sqrt{2}}$	0	1/8	0	1/8	0
r	0	0	1/8	0	0



(c) $p_X(x) = \sum_k p_{XY}(x, k)$

$p_X(r) = 1/8$ $\quad p_Y(r) = 1/8$

$p_X(\frac{r}{\sqrt{2}}) = 1/4$ $\quad p_Y(\frac{r}{\sqrt{2}}) = 1/4$

$p_X(0) = 1/4$ $\quad p_Y(0) = 1/4$

$p_X(\frac{-r}{\sqrt{2}}) = 1/4$ $\quad p_Y(-\frac{r}{\sqrt{2}}) = 1/4$

$p_X(-r) = 1/8$ $\quad p_Y(-r) = 1/8$

(d) $P[A] = p_X(0) = 1/4$

$P[B] = 1 - p_Y(1) = 7/8$

$P[C] = p_{XY}(\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}}) = 1/8$

$P[D] = p_{XY}(-r, 0) = 1/8$

5.19

b) $P[e^X < 6] = P[X < \ln 6] = F_{XY}(\ln 6, \infty) = F_X(\ln 6)$

assuming $P[X = \ln 6] = 0$ so $P[X \le \ln 6] = P[X < \ln 6]$

c) $P[\max(X,Y) < 6] = P[X<6, Y<6] = F_{XY}(6,6)$

assuming $P[X=6]=0$ and $P[Y=6]=0$

i) $P[\max(|X|,|Y|) < 3] = P[|X|<3, |Y|<3]$

$= P[-3<X<3, -3<Y<3]$ square region

$= F_{XY}(3,3) - F_{XY}(3,-3) - F_{XY}(-3,3) + F_{XY}(-3,-3)$

assuming $P[X=3] = P[X=-3] = 0$

$P[Y=3] = P[Y=-3] = 0.$

5.20 a) For $x > 0$, $y > 0$

$$\begin{aligned} F_{XY}(x,y) &= \int_0^x \int_0^y axe^{-ax^2/2} bye^{-by^2/2} dx dy \\ &= (1 - e^{-ax^2/2})(1 - e^{-by^2/2}) \end{aligned}$$

b)

$$\begin{aligned} P[X > Y] &= \int_0^\infty \int_0^x axe^{-ax^2/2} bye^{-by^2/2} dy dx \\ &= \int_0^\infty axe^{-ax^2/2}(1 - e^{-bx^2/2}) dx \\ &= 1 - \frac{a}{a+b} \end{aligned}$$

c)

$$F_X(x) = \lim_{y \to \infty} F_X(x,y) = 1 - e^{-ax^2/2} \qquad x > 0$$

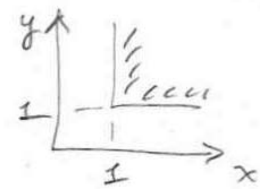
$$\Rightarrow f_X(x) = \frac{d}{dx} F_X(x) = axe^{-ax^2/2} \qquad x > 0$$

Similarly

$$f_Y(y) = bye^{-by^2/2}$$

5.21

$$F_{XY}(x,y) = \begin{cases} 1 - \dfrac{1}{x^2y^2} & x>1,\ y>1 \\ 0 & \text{elsewhere.} \end{cases}$$



$$F_X(x) = \lim_{y\to\infty} F_{XY}(x,y) = 1 \quad \text{all } x>1$$

$F_X(x)$ cannot be equal to 1 for all x

so not a valid cdf

5.4 The Joint cdf of Two Continuous Random Variables

5.24

a) for $x > 0, y > 0$

$$F_{XY}(x,y) = \int_0^x \int_0^y \frac{1}{2} e^{-x/2} 2y e^{-y^2} dx\,dy$$

$$= (1 - e^{-x/2})(1 - e^{-y^2})$$

b)
$$P[X > Y] = \int_0^\infty \int_0^x 2y e^{-y^2} dy \frac{1}{2} e^{-x/2} dx$$

$$= \int_0^\infty (1 - e^{-x^2}) \frac{1}{2} e^{-x/2} dx$$

$$(x + \tfrac{1}{4})^2 = \tfrac{1}{16}$$

$$= 1 - \frac{1}{2} \int_0^\infty e^{-(x^2 + x/2)} dx$$

$$= x^2 + \tfrac{1}{2}x + \tfrac{1}{16} - \tfrac{1}{16}$$

$$= 1 - \frac{1}{2} e^{1/16} \underbrace{\int_0^\infty e^{-(x+\frac{1}{4})^2} dx}_{\sqrt{2\pi \frac{1}{2}} = \sqrt{\pi}}$$

← Gaussian pdf
mean $-1/4$
variance $1/2$

$$= 1 - \frac{\sqrt{\pi}\, e^{1/16}}{2}$$

c)
$$F_X(x) = \lim_{y \to \infty} F_{XY}(x,y) = 1 - e^{-x/2} \qquad x > 0$$

$$f_X(x) = \frac{1}{2} e^{-x/2}$$

$$F_Y(y) = 1 - e^{-y^2}$$

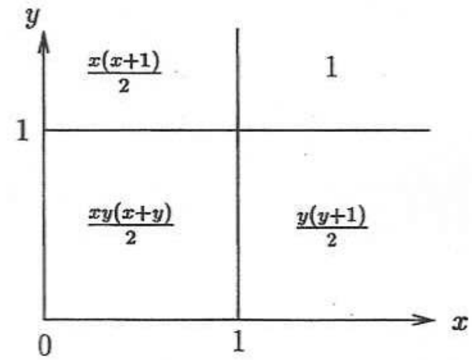
$$f_Y(y) = 2y e^{-y^2}$$

5.25

$$\begin{aligned} 1 &= k\int_0^1\int_0^1 (x+y)dxdy = k\int_0^1 \left(\frac{x^2}{2}+xy\right)_0^1 dy \\ &= k\int_0^1 \left(\frac{1}{2}+y\right)dy = k\left[\frac{1}{2}y+\frac{y^2}{2}\right]_0^1 = k \end{aligned}$$

$\therefore k = 1$

b)

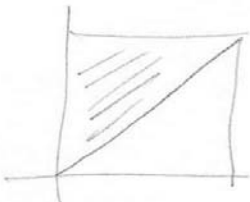


$$\begin{aligned} \text{c)}\quad F_X(x) &= \lim_{y\to\infty} F_{XY}(x,y) = F_{XY}(x,1) \qquad 0 < x < 1 \\ &\Rightarrow f_X(x) = \frac{d}{dx}F_X(x) = x + \frac{1}{2} \end{aligned}$$

Similarly

$$f_Y(y) = y + \frac{1}{2}$$

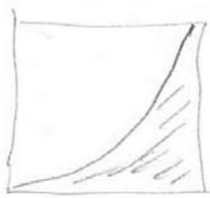
(d) $$P[X<Y] = \int_0^1 dx \int_x^1 (x+y)\,dy = \int_0^1 dx \left[xy + \frac{y^2}{2}\right]_x^1$$



$$= \int_0^1 \left[x + \frac{1}{2} - x^2 - \frac{x^2}{2}\right] dx = \left[\frac{x^2}{2} + \frac{1}{2}x - \frac{3}{2}\frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{2} + \frac{1}{2} - \frac{1}{2} = \frac{1}{2}.$$

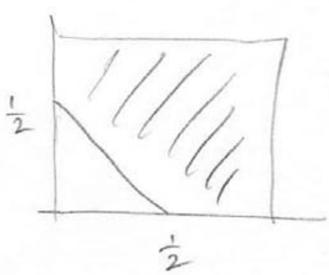
$$P[Y<X^2] = \int_0^1 dx \int_0^{x^2} (x+y)\,dy = \int_0^1 dx \left[xy + \frac{y^2}{2}\right]_0^{x^2}$$



$$= \int_0^1 \left[x^3 + \frac{x^4}{2}\right] dx = \left[\frac{x^4}{4} + \frac{1}{2}\frac{x^5}{5}\right]_0^1$$

$$= \frac{1}{4} + \frac{1}{10} = \frac{7}{20}$$

$$P\left[X+Y>\frac{1}{2}\right] = 1 - \int_0^{\frac{1}{2}} dx \int_0^{\frac{1}{2}-x} (x+y)\,dy$$



$$= 1 - \int_0^{\frac{1}{2}} dx \left[xy + \frac{y^2}{2}\right]_0^{\frac{1}{2}-x}$$

$$= 1 - \int_0^{\frac{1}{2}} dx \left[x\left(\frac{1}{2}-x\right) + \frac{1}{2}\left(\frac{1}{2}-x\right)^2\right]$$

$$= 1 - \int_0^{\frac{1}{2}} \Big[\underbrace{\frac{1}{2}x - x^2 + \frac{1}{2}\left(\frac{1}{4} - x + x^2\right)}_{\left[\frac{1}{8} - \frac{1}{2}x^2\right]}\Big] dx$$

$$= 1 - \left[\frac{1}{8}x - \frac{1}{2}\frac{x^3}{3}\right]_0^{1/2}$$

$$= 1 - \Big[\underbrace{\frac{1}{8}\left(\frac{1}{2}\right)}_{\frac{1}{8}\left(\frac{2}{4}\right)} - \frac{1}{6}\frac{1}{8}\Big] = \frac{46}{48}$$

5.28

~~4.10~~

$$f_X(x) = \int_0^\infty x e^{-x} e^{-xy} dy = x e^{-x}\left(\frac{-1}{x}e^{-xy}\right)_0^\infty = e^{-x}$$

$$f_Y(y) = \int_0^\infty x e^{-x(1+y)} dx = \frac{e^{-x(1+y)}((1+y)x-1)}{(1+y)^2}\Bigg|_0^\infty = \frac{1}{(1+y)^2}$$

5.29

(a) $f_{R,\Theta}(r,\theta) = \frac{\partial^2}{\partial r \partial \theta} F_{R,\Theta}(r,\theta) = \frac{\partial^2}{\partial r \partial \theta} r^2 \frac{\theta}{2\pi}$ $\quad 0 \le r \le 1$, $0 \le \theta \le 2\pi$

$= \frac{\partial}{\partial r} r^2 \frac{1}{2\pi} = (2r)\left(\frac{1}{2\pi}\right)$ $\quad 0 \le r \le 1$, $0 \le \theta \le 2\pi$

(b) $f_R(r) = \int_0^{2\pi} 2r \frac{1}{2\pi} d\theta = 2r$ $\quad 0 \le r \le 1$

$f_\Theta(\theta) = \int_0^1 2r \frac{1}{2\pi} d\theta = \frac{1}{2\pi} r^2\Big|_0^1 = \frac{1}{2\pi}$

$0 \le \theta \le 2\pi$

5.30

~~4.14~~

$$P[X^2+Y^2<R^2] = \iint_{x^2+y^2<R^2} \frac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/2\sigma^2} dx dy$$

$$= \int_0^{2\pi}\int_0^R \frac{1}{2\pi\sigma^2} e^{-r^2/2\sigma^2} r dr d\theta$$

where we let $x = r\cos\theta$, $y = r\sin\theta$

$$= \frac{r}{\sigma^2}\int_0^R r e^{-r^2/2\sigma^2} dr$$

$$= 1 - e^{-R^2/2\sigma^2}$$