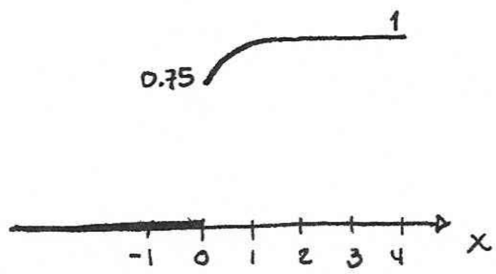
4.11

a)



Mixed type random variable

b) $P[X \leq 2] = 1 - \frac{1}{4}e^{-(2)}$

$= 0.9662$

$P[X = 0] = 1 - \frac{1}{4}e^{-(0)}$

$= 0.75$

$P[X < 0] = 0$

$P[2 < X < 6] = P[X \leq 6] - P[X \leq 2]$

$= 1 - \frac{1}{4}e^{-(6)} - 1 + \frac{1}{4}e^{-(2)}$

$= 0.0332$

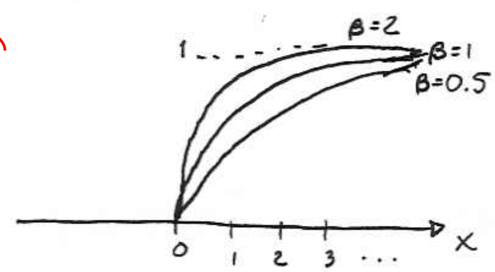
$P[X > 10] = 1 - P[X \leq 10]$

$= 1 - \left(1 - \frac{1}{4}e^{-(10)}\right)$

$= 1.135 \times 10^{-5}$

4.13

a)



b) $P[j\lambda < X < (j+1)\lambda] = P[X \le (j+1)\lambda] - P[X \le j\lambda]$

$$= 1 - e^{-\left(\frac{(j+1)\lambda}{\lambda}\right)^\beta} - 1 + e^{-\left(\frac{j\lambda}{\lambda}\right)^\beta} \qquad j \ge 0$$

$$= e^{-j^\beta} - e^{-(j+1)^\beta} \qquad j \ge 0$$

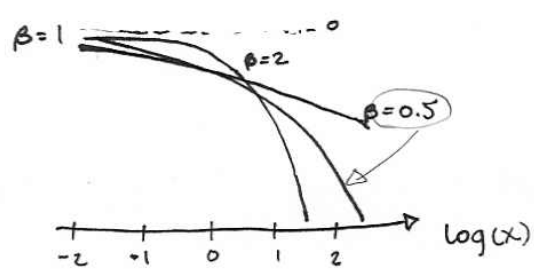
$$= \begin{cases} 0 & j < -1 \\ 1 - e^{-(j+1)^\beta} & -1 \le j < 0 \\ e^{-j^\beta} - e^{-(j+1)^\beta} & j \ge -1 \end{cases}$$

$$P[X > j\lambda] = 1 - P[X \le j\lambda] = 1 - \left(1 - e^{-\left(\frac{j\lambda}{\lambda}\right)^\beta}\right) \qquad j \ge 0$$

$$= \begin{cases} 0 & j < 0 \\ e^{-j^\beta} & j \ge 0 \end{cases}$$

c) $\log P[X > x] = \begin{cases} \operatorname{Log}(1) & x < 0 \\ \operatorname{Log}\left(e^{-\left(\frac{x}{\lambda}\right)^\beta}\right) & x \ge 0 \end{cases}$

$$= \begin{cases} 0 & x < 0 \\ -\left(\frac{x}{\lambda}\right)^\beta & x \ge 0 \end{cases}$$



4.2 The Probability Density Function

4.14

$$1 = \int_{-1}^{1} f_X(x)dx = c\int_{-1}^{1} x(1-x^4)dx = c\left[x - \frac{x^5}{5}\right]_{-1}^{1} = \frac{8}{5}c \Rightarrow c = \frac{5}{8}$$

b) $F_X(x) = 0$ for $x < -1$; $F_X(x) = 1$ for $x > 1$
For $-1 \le x \le 1$

$$F_X(x) = \int_{-1}^{x} \frac{5}{8}(1 - x'^4)dx' = \frac{1}{2} + \frac{1}{8}(5x - x^5)$$

c) $P[|X| < \frac{1}{2}] = P[-\frac{1}{2} < X < \frac{1}{2}] = F_X(\frac{1}{2}) - F_X(-\frac{1}{2}) = \frac{79}{8(16)}$

4.15

a) We use the fact that the pdf must integrate to one:

$$1 = \int_0^1 f_X(x)dx = c\int_0^1 x(1-x)dx = c\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{c}{6}$$

$$\Rightarrow \quad c = 6$$

b) $P\left[\frac{1}{2} \le X \le \frac{3}{4}\right] = 6\int_{1/2}^{3/4} x(1-x)dx = 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{1/2}^{3/4} = 0.34375$

c) For $x < 0$, $F_X(x) = 0$; for $x > 1$, $F_X(x) = 1$
For $0 \le x \le 1$

$$F_X(x) = \int_0^x f_X(x')dx' = 3x^2 - 2x^3$$

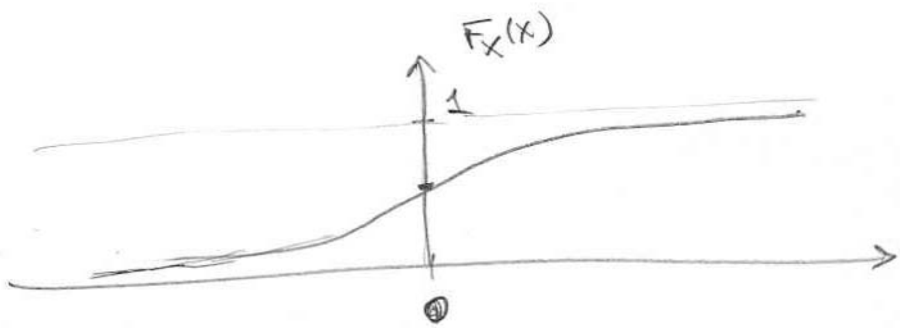
4.23

$$f_X(x) = \frac{\alpha/\pi}{x^2+\alpha^2}$$

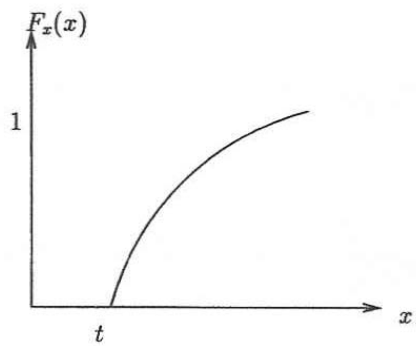
$$F_X(x) = \int_{-\infty}^{x} \frac{\alpha/\pi}{t^2+\alpha^2}\,dt = \int_{-\infty}^{x} \frac{1/\pi}{1+\left(\frac{t}{\alpha}\right)^2}\,d\left(\frac{t}{\alpha}\right)$$

$$= \frac{1}{\pi}\tan^{-1}\left(\frac{t}{\alpha}\right)\Big|_{-\infty}^{x}$$

$$= \frac{1}{\pi}\left[\tan^{-1}\left(\frac{x}{\alpha}\right) + \frac{\pi}{2}\right] \qquad -\infty < x < \infty$$



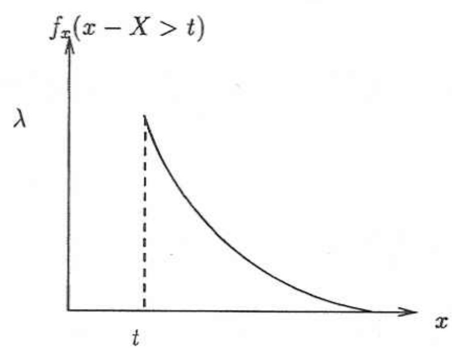
4.29 a)



$$
\begin{aligned}
F_X(x|X>T) &= \frac{P[\{X \le x\} \cap \{X > t\}]}{P[X>t]} \\
&= \begin{cases} 0 & x < t \\ \frac{F_X(x)-F_X(t)}{1-F_X(t)} & x \ge t \end{cases} \\
&= \begin{cases} 0 & x < t \\ \frac{(1-e^{-\lambda x})-(1-e^{-\lambda t})}{1-(1-e^{-\lambda t})} & x \ge t \end{cases} \\
&= \begin{cases} 0 & x < t \\ \frac{e^{-\lambda x}-e^{-\lambda t}}{e^{-\lambda t}} & x \ge t \end{cases}
\end{aligned}
$$

$F_x(x|x>t)$ is delayed version of $F_x(x)$

b) $f_x(x|x>t) = \dfrac{f_x(x)}{1-F_x(t)} = \dfrac{\lambda e^{-\lambda x}}{e^{-\lambda t}} = \lambda e^{-\lambda(x-t)}, \quad x \ge t$



c)

$$
\begin{aligned}
P &= [X > t + x | X > t] \qquad x \ge 0 \\
&= \frac{P[\{X > t+x\} \cap \{X > t\}]}{P[X>t]} \\
&= \frac{1-F_X(t+x)}{1-F_X(t)} \\
&= \frac{1-(1-e^{-\lambda(t+x)})}{1-(1-e^{-\lambda t})} \\
&= e^{-\lambda x} \\
&= P[X > x]
\end{aligned}
$$

Additional waiting time does not depend on time already spent waiting ⇒ "memoryless"

4.49

$$
\begin{aligned}
Y &= A\cos \omega t + c \\
E[Y] &= E[A]\cos \omega t + c = m\cos \omega t + c \\
VAR[Y] &= \cos^2 \omega t\, VAR[A] = \sigma^2 \cos^2 \omega t
\end{aligned}
$$

4.50 a)

$$
\begin{aligned}
\mathcal{E}[Y] &= \int_{-\infty}^{\infty} g(x) f_X(x)dx \quad \text{write integral into three parts} \\
&= -a\int_{-\infty}^{-a} f_X(x)dx + \int_{-a}^{a} x f_X(x)dx + a\int_{a}^{\infty} f_X(x)dx \\
&= -aF_X(-a) + \int_{-a}^{a} x f_X(x)dx + a(1 - F_X(a^-)) \\
\mathcal{E}[Y^2] &= a^2 F_X(-a) + \int_{-a}^{a} x^2 f_X(x)dx + a^2(1 - F_X(a^-)) \\
VAR[Y] &= \mathcal{E}[Y^2] - \mathcal{E}[Y]^2
\end{aligned}
$$

4.50 b)

$$
E[Y] = \underbrace{-(1)\underbrace{P[Y\le -1]}_{\frac{1}{2}e^{-1}} + (1)\underbrace{P[Y\ge 1]}_{\frac{1}{2}e^{-1}}}_{=0} + \underbrace{\int_{-1}^{1} \underbrace{x}_{\text{odd}}\, \underbrace{\tfrac{1}{2}e^{-|x|}}_{\text{even}}\, dx}_{=0} = 0
$$

$$
VAR[Y] = E[Y^2] = (-1)^2 P[Y\le -1] + (1)^2 P[Y\ge 1] + \int_{-1}^{1} x^2 \tfrac{1}{2} e^{-|x|} dx
$$

$$
= e^{-1} + 2\cdot\tfrac{1}{2}\underbrace{\int_0^1 x^2 e^{-x} dx}_{e^{-x}(x^2+2x+2)\Big|_0^1} \quad \text{from appendix}
$$

$$
= e^{-1} + 5e^{-1} - 2 = 6e^{-1} - 2
$$

4.52

a) $E[Y] = 2E[X] + 2$

$VAR[Y] = VAR[2X+2] = VAR[2X] = 4VAR[X]$

b) Laplacian R.V. $E[X] = 0$

$VAR[X] = \frac{2}{\alpha^2}$

$E[Y] = 2$

$VAR[Y] = 4\left(\frac{2}{\alpha^2}\right) = \frac{8}{\alpha^2}$

c) Gaussian R.V. $E[X] = m$

$VAR[X] = \sigma^2$

$E[Y] = 2m + 2$

$VAR[Y] = 4\sigma^2$

d) $E[X] = b\int_0^1 \cos(2\pi u)\,du = -b\sin(2\pi u)\Big|_0^1 = 0$

$$\begin{aligned}VAR[X] &= b^2\int_0^1 \cos^2(2\pi u)\,du \\ &= b^2\int_0^1 \frac{1}{2}du + \frac{b^2}{2}\int_0^1 \cos 4\pi u\,du \\ &= b^2\frac{1}{2} + b^2\left(\frac{1}{4\pi}\right)(-\sin 4\pi u)\Big|_0^1 \\ &= \frac{b^2}{2}\end{aligned}$$

$E[Y] = 2$

$VAR[Y] = \frac{4b^2}{2}$

4.53

$$\begin{aligned}\mathcal{E}[X^n] &= \int_0^1 x^n dx = \frac{x^{n+1}}{n+1}\bigg|_0^1 = \frac{1}{n+1} \\ \mathcal{E}[Y^n] &= \frac{1}{b-a}\int_a^b y^n dy = \frac{1}{b-a}\left[\frac{b^{n+1}-a^{n+1}}{n+1}\right]\end{aligned}$$

4.65) Gamma RV

a) $$E[X] = \int_0^\infty x \frac{\lambda(\lambda x)^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} dx = \int_0^\infty \frac{\lambda(\lambda x)^{\alpha} e^{-\lambda x}}{\lambda\,\Gamma(\alpha)} dx$$

$$= \int_0^\infty \frac{\Gamma(\alpha+1)}{\lambda\Gamma(\alpha)} \frac{\lambda(\lambda x)^{\alpha}}{\Gamma(\alpha+1)} e^{-\lambda x} dx = \frac{\Gamma(\alpha+1)/\Gamma(\alpha)}{\lambda} \underbrace{\int_0^\infty \frac{\lambda(\lambda x)^{\alpha}}{\Gamma(\alpha+1)} e^{-\lambda x} dx}_{1}$$

$$= \frac{\alpha}{\lambda}$$

b) $$E[X^2] = \int_0^\infty x^2 \frac{\lambda(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda x} dx = \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)} \frac{1}{\lambda^2} \underbrace{\int_0^\infty \frac{\lambda(\lambda x)^{\alpha+1}}{\Gamma(\alpha+2)} e^{-\lambda x} dx}_{1}$$

$$= \frac{(\alpha+1)(\alpha)}{\lambda^2}$$

$$\Gamma(\alpha+2) = (\alpha+1)\Gamma(\alpha+1) = (\alpha+1)\alpha\Gamma(\alpha)$$

$$VAR[X] = \frac{(\alpha+1)\alpha}{\lambda^2} - \frac{\alpha^2}{\lambda^2}$$

$$= \frac{\alpha}{\lambda^2}$$

c) m-Erlang $\alpha = m$

$$E[X] = \frac{m}{\lambda} \qquad VAR[X] = \frac{m}{\lambda^2}$$

d) chi-square $\alpha = k/2 \quad \lambda = \frac{1}{2}$

$$E[X] = \frac{k}{2}\frac{1}{1/2} = k$$

$$VAR[X] = \frac{k}{2}\frac{1}{1/4} = 2k$$

4.73

$$P[Z \le z] = P[X^2 \le z] = P[X \le \sqrt{z}]$$ since X is nonnegative

$$f_Z(z) = \frac{d}{dx} F_X(\sqrt{z}) = f_X(\sqrt{z})\, \frac{1}{2} z^{-1/2}$$

$$= \frac{f_X(\sqrt{z})}{2\sqrt{z}} = \frac{\sqrt{z}}{2\sqrt{z}} \frac{1}{\alpha^2} e^{-\sqrt{z}^2/2\alpha^2} = \frac{1}{2\alpha^2} e^{-z/2\alpha^2}$$

exponential RV.

4.74

$$P[N=0] = F_X(\pi)$$

$$P[N=1] = F_X(2\pi) - F_X(\pi)$$

$$\vdots$$

$$P[N=n] = F_X((n+1)\pi) - F_X(n\pi)$$

$$= 1 - e^{-\lambda(n+1)\pi} - (1 - e^{-\lambda n\pi})$$

$$= (e^{-\lambda n\pi} - e^{-\lambda(n+1)\pi})$$

$$= e^{-\lambda n\pi}(1 - e^{-\lambda\pi})$$ $\lambda = \frac{1}{5\pi}$

$$= e^{-\frac{n\pi}{5\pi}}(1 - e^{-\frac{\pi}{5\pi}})$$

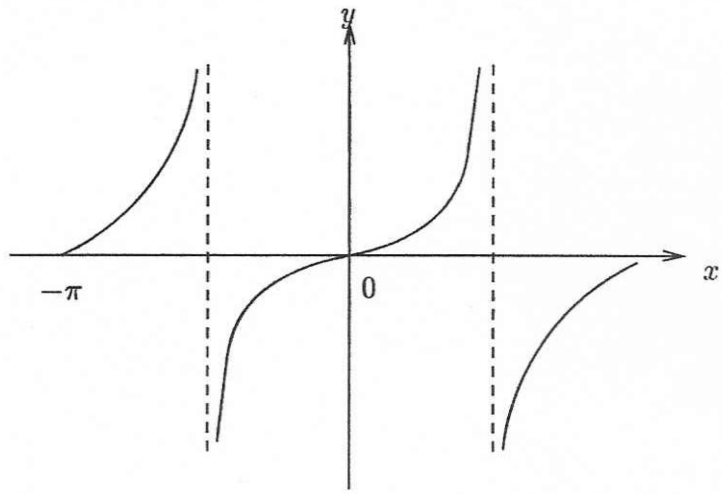
$$= (1 - e^{-1/5})(e^{-1/5})^n \qquad n = 0, 1, 2, \ldots$$

geometric RV

4.87 $Y = a \tan X$.

$$x = \tan^{-1}(y/a), \; -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

$$\frac{dx}{dy} = \frac{1}{1 + (y/a)^2}\frac{1}{a} = \frac{a}{y^2 + a^2}$$



$$\begin{aligned} f_X(y) &= \sum_k f_X(x) \left|\frac{dx}{dy}\right| \Big|_{x=x_k} \\ &= 2 \cdot \frac{1}{2\pi}\frac{a}{y^2 + a^2} \\ &= \frac{a/\pi}{y^2 + a^2} \end{aligned}$$

Y is a Cauchy RV.

4.88

$X = -\ln(1-U)$

$e^{-x} = (1-U)$

$U = 1 - e^{-x}$

$f_X(x) = f_U(U)\left|\frac{dU}{dx}\right|$

$= f_U(1-e^{-x})\,|e^{-x}|$

$= e^{-x}$