3.14 (a) $1 = \sum_{k=1}^{\infty} \frac{c}{k^2} = c \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$ is a special case of the zeta function

$= 1.6449 \Rightarrow c = 0.608$

The sum of the first 100 terms gives $1.6349 \Rightarrow c \approx 0.611$

(b) $P[X > 6] = 1 - P[X \le 5] = 1 - c\underbrace{\left[1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}\right]}_{0.88979}$

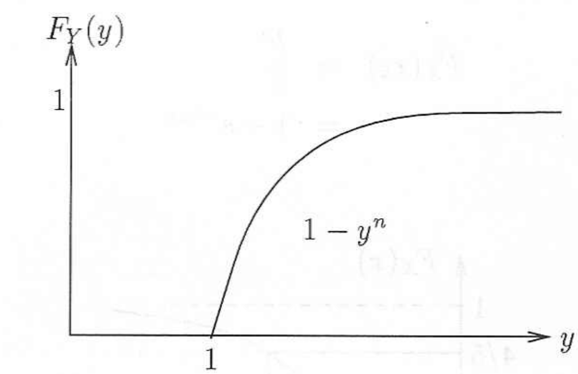
$= 0.1102$

(c) $P[4 \le X \le 8] = c\left[\frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \frac{1}{49} + \frac{1}{64}\right] = 0.1011$

3.15 $P[X \ge 5] = \sum_{k=5}^{7} p_k = \frac{3}{8}$

$P[Y \ge 5] = \sum_{k=5}^{7} p'_k = 3 \cdot \frac{3}{16} = \frac{9}{16}$

3.16 a)



b) $P[k < Y \le k+1] = F_Y(k+1) - F_Y(k) = \left(\frac{1}{k}\right)^n - \left(\frac{1}{k+1}\right)^n$

3.21

2nd toss

1st toss	1	2	3	4	5	6
1	0	1	2	3	4	5
2	-1	0	1	2	3	4
3	-2	-1	0	1	2	3
4	-3	-2	-1	0	1	2
5	-4	-3	-2	-1	0	1
6	-5	-4	-3	-2	-1	0

$P[X=0] = \frac{6}{36}$

$P[X=1] = \frac{5}{36} = P[X=-1]$

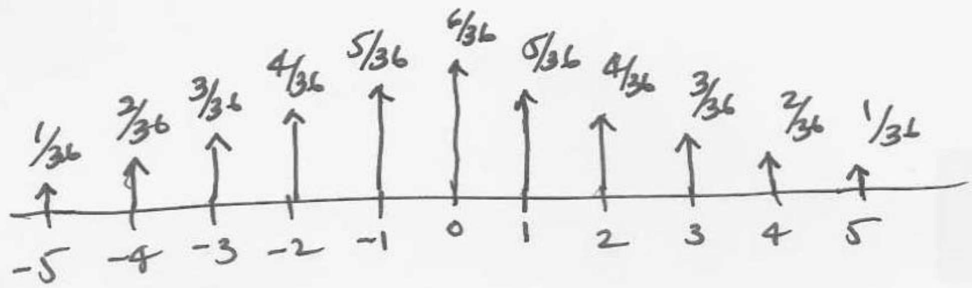
$P[X=2] = \frac{4}{36} = P[X=-2]$

$P[X=3] = \frac{3}{36} = P[X=-3]$

$P[X=4] = \frac{2}{36} = P[X=-4]$

$P[X=5] = \frac{1}{36} = P[X=-5]$

$$P[X=k] = \frac{6-|k|}{36}, \quad |k| \le 5$$



$$P[|X| \le 2] = \frac{6}{36} + 2\frac{5}{36} + 2 \cdot \frac{4}{36} = \frac{24}{36}$$

3.28

$$E[X] = \sum_{j=1}^{\infty} j\, P[X=j] = \sum_{j=1}^{\infty} j \frac{c}{j^2} = c\sum_{j=1}^{\infty} \frac{1}{j} = \infty$$

mean does not exist.

$$E[X^2] = \sum_{j=1}^{\infty} j^2 \frac{c}{j^2} = c\sum_{j=1}^{\infty} 1 = \infty$$

none of the moments exist

This pmf decays sufficiently fast that probabilities add to 1, but too slowly for moments to exist.

3.29

$$E[Y] = -1\cdot\frac{1}{10} + 0\cdot\frac{2}{10} + 1\cdot\frac{3}{10} + 2\cdot\frac{4}{10} = \frac{10}{10} = 1$$

$$E[Y^2] = 1\cdot\frac{1}{10} + 1\cdot\frac{3}{10} + 4\cdot\frac{4}{10} = \frac{20}{10} = 2$$

$$VAR[Y] = 2 - 1^2 = 1.$$

3.48

(a) $A_i = \{U_i < 0.25\}$

$A_i^c = \{U_i > 0.25\}$

$P[\overbrace{A_1 A_2 A_3 A_4 A_5^c A_6^c A_7^c A_8^c}^{\text{intersections}}] = (0.25)^4 (0.75)^4$

$= 0.00124$

(b) $P[N=4] = \binom{8}{4}(0.25)^4 (0.75)^4 = 0.0865$

(c) $A_i = \{U_i < 0.25\}$

$B_i = \{0.25 < U_i < 0.75\}$

$C_i = \{U_i > 0.75\}$

$P[A_1 A_2 A_3 B_4 B_5 C_6 C_7 C_8] = (0.25)^3 (0.5)^2 (0.25)^3$

$= (0.25)^6 (0.5)^2$

$= 6.10 \times 10^{-5}$

(d) $P[N_1 = 3, N_2 = 2, N_3 = 3] = \dfrac{8!}{3!\,2!\,3!}\, 0.25^3\ 0.5^2\ 0.25^3$

multinomial

$= 0.0342$

(e) $P[A_1 A_2 A_3 A_4 C_5 C_6 C_7 C_8] = (0.25)^4 (0.25)^4 = 1.526 \times 10^{-5}$

(f) $P[N_1 = 4, N_2 = 0, N_3 = 4] = \dfrac{8!}{4!\,0!\,4!} (0.25)^4 (0.5)^0 (0.25)^4$

$= 0.00107$

3.49

This Octave program will plot binomial pmf.

```
> n=4;
> x=[0:n];
> p=0.10;
> stem(binomial_pdf(x,n,p))
```

3.51 a)

$$\frac{p_k}{p_{k-1}} = \frac{\binom{n}{k} p^k q^{n-k}}{\binom{n}{k-1} p^{k-1} q^{n-k+1}} = \frac{\frac{n!}{k!(n-k)!}}{\frac{n!}{(k-1)!}} \frac{p}{q} = \frac{(n-k+1)p}{kq}$$

$$= \frac{(n+1)p - k(1-q)}{kq} = 1 + \frac{(n+1)p-k}{kq}$$

b) First suppose $(n+1)p$ is not an integer, then
for $0 \leq k \leq [(n+1)p] < (n+1)p$

$$(n+1)p - k > 0$$

so

$$\frac{p_k}{p_{k-1}} = 1 + \frac{(n+1)p-k}{kq} > 1$$

$\Rightarrow p_k$ increases as k increases from 0 to $[(n+1)p]$
for $k > (n+1)p \geq [(n+1)p]$

$$(n+1)p - k < 0$$

so

$$\frac{p_k}{p_{k-1}} = 1 + \frac{(n+1)p-k}{kq} < 1$$

$\Rightarrow p_k$ decreases as k increases beyond $[(n+1)p]$
$\therefore p_k$ attains its maximum at $k_{MAX} = [(n+1)p]$
If $(n+1)p = k_{MAX}$ then above implies that

$$\frac{p_{k_{MAX}}}{p_{k_{MAX}-1}} = 1 \Rightarrow p_{k_{MAX}} = p_{k_{MAX}} - 1$$

3.54 N geometric $n = 1, 2, \ldots$

(a) $$P[N=k \mid N \le m] = \frac{P[N=k, N\le m]}{P[N\le m]} = \frac{P[N=k]}{P[N\le m]} \quad 1 \le k \le m$$

$$= \frac{p(1-p)^{k-1}}{\sum_{j=1}^{m} p(1-p)^{j-1}} = \frac{p(1-p)^{k-1}}{1-(1-p)^m} \quad 1 \le k \le m$$

(b) $$P[N \text{ is even}] = \sum_{j=0}^{\infty} p(1-p)^{2j} = p \sum_{j=0}^{\infty} ((1-p)^2)^j$$

$$= \frac{p}{1-(1-p)^2}$$

3.55 $$P[M \ge k+j \mid M > j] = \frac{P[M \ge k+j, M > j]}{P[M > j]} = \frac{P[M \ge k+j]}{P[M > j]} \quad \text{for } k \ge 1$$

$$= \frac{\sum_{i=k+j}^{\infty} p(1-p)^{i-1}}{\sum_{i'=j+1}^{\infty} p(1-p)^{i'-1}}$$

$$= \frac{(1-p)^{k+j-1}}{(1-p)^j} = (1-p)^{k-1} = P[M \ge k]$$

The probability of k additional trials until the first success is independent of how many failures have already transpired.

3.67 $n = 10^4$ drives $\quad p = 10^{-3} \quad np = 10^4(10^{-3}) = 10/\text{day}$

(a) $P[N=0] \simeq e^{-np} = e^{-10} = 4.54 \times 10^{-5}$

(b) Failure rate in 2 days = 20

$$P[N \leq 10] = \sum_{j=0}^{10} \frac{(20)^j}{j!} e^{-20} = 1.08 \times 10^{-2}$$

(c) $0.99 = P[N \leq k] = \sum_{j=0}^{k} \frac{10^j}{j!} e^{-10} \Rightarrow P[N \leq$ [redacted] $] = 0.986$

3.68 $p = 10^{-5} \quad n = 10^4 \quad np = 10^{-1}$

(a) $P[N=0] = e^{-np} = 0.9048$

$$P[N \leq 3] = \sum_{k=0}^{3} \frac{(0.01)^k}{k!} e^{-np} = 0.9999961$$

(b) Find p so that

$$0.99 = P[N \geq 1] = 1 - P[N=0] = 1 - e^{-np}$$

$$0.01 = e^{-np}$$

$$\Rightarrow p = \frac{\ln 100}{n} = 4.6 \times 10^{-6}$$

3.69

65
$$\mathcal{E}[X] = \sum_{k=1}^{n} kP[X=k] = \sum_{k=1}^{n} \frac{k}{n} = \frac{1}{n}\sum_{k=1}^{n} k = \frac{n(n+1)}{2n} = \frac{n+1}{2}$$

$$\sigma_X^2 = \mathcal{E}[X^2] - \mathcal{E}[X]^2 = \sum_{k=1}^{n} \frac{k^2}{n} - \left(\frac{n+1}{2}\right)^2 = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4}$$

$$= \frac{n^2-1}{12}$$

3.70 X uniform in $\{-3,-2,\dots,3,4\}$ $\quad P[X=j] = \frac{1}{8}$

(a) $E[X] = -4 + \frac{8+1}{2} = 0.5$

$VAR(X) = \frac{8^2-1}{12} = \frac{63}{12} = \frac{21}{4}$

(b) $E[Y] = E[-2X^2+3] = -2E[X^2]+3$

$= -2[VAR(X) + E[X]^2] + 3$

$= -2\left[\frac{21}{4} + (0.5)^2\right] + 3 = -8$

$E[Y^2] = E[(-2X^2+3)^2] = E[4X^4 - 12X^2 + 9]$

$VAR[Y] = E[Y^2] - E[Y]^2$

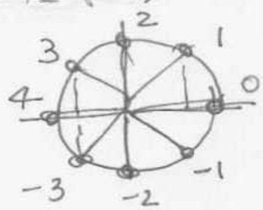
$= 4E[X^4] - 12E[X^2] + 9 - (-8)^2$

$E[X^4] = \frac{1}{8}\left[(-3)^2 + (-2)^2 + (-1)^2 + 0^2 + 1^2 + 2^2 + 3^2 + 4^2\right]$

$= \frac{44}{8} = \frac{11}{2}$

$VAR[Y] = 4\left(\frac{11}{2}\right) - 12(-11) + 9 - 64 = 99$

(c) $W = \cos\left(\frac{\pi X}{8}\right)$



x	-3	-2	-1	0	1	2	3	4
w	$-\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	1	$\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$	-1

$P\left[W = -\frac{1}{\sqrt{2}}\right] = \frac{2}{8} \quad P[W=0] = \frac{2}{8}$

$P\left[W = \frac{1}{\sqrt{2}}\right] = \frac{2}{8} \quad P[W=1] = \frac{1}{8} \quad P[W=-1] = \frac{1}{8}$