Chapter 2: Basic Concepts of Probability Theory

2.1 Specifying Random Experiments

2.1 a. $S = \{1, 2, 3, 4, 5, 6\}$

b. $A = \{2, 4, 6\}$

c. $A^c = \{1, 3, 5\}$ "odd number of dots"

2.3 a) $\mathcal{S} = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

b) $A = \{2, 4, 6, 8, 10, 12\}$

c)
$$\begin{array}{lll} \{\text{sum} = 2\} & \text{corresponds to} & \{(1,1)\} \\ \{\text{sum} = 3\} & \text{corresponds to} & \{(1,2),(2,1)\} \\ \{\text{sum} = 4\} & \text{corresponds to} & \{(1,3),(2,2),(3,1)\} \end{array}$$

and in general for $1 < k \le 12$

$$\{\text{sum} = k\} = \bigcup_{j=1}^{\min(6,k-1)} \{j, k-j)\}$$

2.4

(a)

X \ Y	-2	-1	0	1	2
+2	—	—	(2,0)	(2,1)	(2,2)
-2	(-2,-2)	(-2,-1)	(-2,0)	—	—

(b) "X definitely +2" (based on observed Y): $\{(2,1),(2,2)\}$

(c) $\{Y=0\} = \{(2,0), (-2,0)\}$

"observed output is zero"
cannot determine input

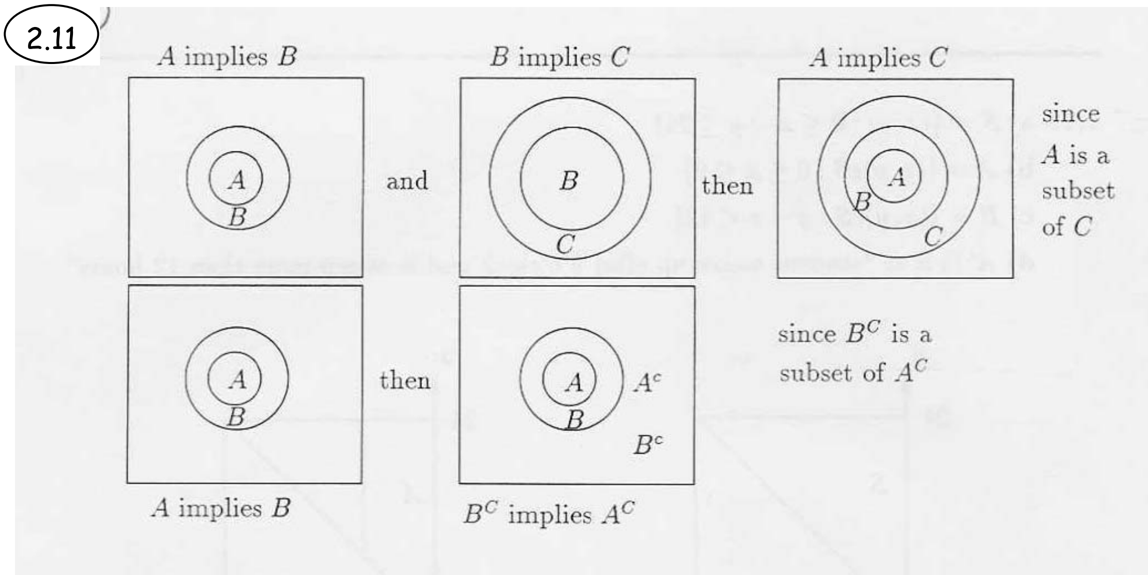
2.5 a) Each testing of a pen has two possible outcomes: "pen good" (g) or "pen bad" (b). The experiment consists of testing pens until a good pen is found. Therefore each outcome of the experiment consists of a string of "b's" ended by a "g". We assume that each pen is not put back in the drawer after being test. Thus $\mathcal{S} = \{g, bg, bbg, bbbg\}$

b) We now simply record the number of pens tested, so $\mathcal{S} = \{1, 2, 3, 4\}$

c) The outcome now consists of a substring of "b's" and one "g" in any order followed by a final "g". $\mathcal{S} = \{gg, bgg, gbg, gbbg, bbgg, gbbbg, bgbbg, bbgbg, bbbgg\}$

d) $\mathcal{S} = \{2, 3, 4, 5\}$

2.11



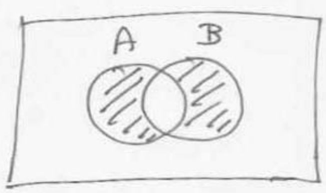
2.12 Given $A \cup B = A$ and $A \cap B = A$ claim $A = B$

Let $\xi \in A$, then $\xi \in A \cap B \Rightarrow \xi \in B$ $\therefore A \subset B$

Let $\xi \in B$ then $\xi \in A \cup B \Rightarrow \xi \in A$ $\therefore B \subset A$

$\therefore A = B$.

2.13 "Exactly one of A or B occurs" $= (A \cap B^c) \cup (A^c \cap B)$



2.14 a) $(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C)$

b) $(A \cap B \cap C^c) \cup (A \cap B^c \cap C) \cup (A^c \cap B \cap C)$

c) $A \cup B \cup C$

d) $(A^c \cap B^c \cap C^c) \cup (A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C)$

e) $A^c \cap B^c \cap C^c$

2.15 ⓐ $D = A_1 \cap A_2 \cap A_3$

ⓑ $D = A_1 \cup A_2 \cup A_3$

ⓒ $D = (A_1 \cap A_2 \cap A_3) \cup (A_1^c \cap A_2 \cap A_3) \cup (A_1 \cap A_2^c \cap A_3) \cup (A_1 \cap A_2 \cap A_3^c)$

2.16 ⓐ "System j is up" $= A_{j1} \cap A_{j2}$

"System is up" $= (A_{11} \cap A_{12}) \cup (A_{21} \cap A_{22}) \cup (A_{31} \cap A_{32})$

ⓑ "jth level connection active" if $A_{j1} \cap A_{j2}$

"connection active" if any of 3 connections is active

2.17 a) $\mathcal{S} = \{(x, y) : 0 \le x < y \le 24\}$

b) $A = \{\text{wake up before } 9\} \cap \{\text{to sleep after } 9\}$
$= \{(x, y) \epsilon \mathcal{S} : 0 \le x < 9, \quad 9 < y \le 24\}$

c) $B = \{(x, y) \epsilon \mathcal{S} : y - x < 12\}$

d) $A^c \cap B =$ "student is asleep at 9 o'clock and is asleep more than 12 hours"

2.2 The Axioms of Probability

2.21

a) The sample space in tossing a die is $S = \{1, 2, 3, 4, 5, 6\}$. Let $p_i = P[\{i\}] = p$ since all faces are equally likely. By Axiom 1

$$\begin{aligned} 1 &= P[S] \\ &= P[\{1\} \cup \{2\} \cup \{3\} \cup \{4\} \cup \{5\} \cup \{6\}] \end{aligned}$$

The elementary events $\{i\}$ are mutually exclusive so by Corollary 4:

$$1 = p_1 + p_2 + \ldots + p_6 = 6p \Rightarrow p_i = p = \tfrac{1}{6} \text{ for } i = 1, \ldots, 6$$

2.21

b) $P[A] = P[>4 \text{ dots}] = P[\{5,6\}] = P[\{5\}] + P[\{6\}] = \frac{2}{6}$

$P[B] = P[\text{even } \#] = P[\{2,4,6\}] = P[\{2\}] + P[\{4\}] + P[\{6\}] = \frac{3}{6}$

c) $P[A \cup B] = P[\{2,4,5,6\}] = \frac{4}{6}$

$P[A \cap B] = P[\{6\}] = \frac{1}{6}$

$P[A^c] = 1 - P[A] = \frac{4}{6}$

2.22

a) In first toss, each face occurs with relative frequency 1/6

Each first toss outcome is followed by each possible face 1/6 of the time

∴ Each pair occurs with relative frequency $1/6 \times 1/6 = 1/36$.

b) $P[A] = \frac{21}{36}$ $\quad P[B] = \frac{6}{36}$ $\quad P[C] = \frac{8}{36}$ $\quad P[A \cap B^c] = \frac{15}{36}$ $\quad P[A^c] = \frac{15}{36}$

2.23

$$\begin{aligned} P[\{a, c\}] &= P[\{a\}] + P[\{c\}] = \frac{5}{8} \\ P[\{b, c\}] &= P[\{b\}] + P[\{c\}] = \frac{7}{8} \\ P[\{a, b, c\}] &= P[s] = 1 = P[\{a\}] + P[\{b\} + P[\{c\}] \\ &\Rightarrow P[\{a\}] = \frac{1}{8}, \quad P[\{b\}]\} = \frac{3}{8}, \quad P[\{c\}] = \frac{4}{8} \end{aligned}$$

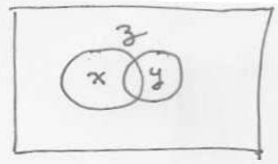
2.24 (a) $P[A \cap B^c] = P[A] - P[A \cap B]$

$P[A^c \cap B] = P[B] - P[A \cap B]$

(b) $P[A \cap B^c \cup A^c \cap B] = P[A] + P[B] - 2P[A \cap B]$

(c) $P[(A \cup B)^c] = 1 - P[A \cup B] = 1 - P[A] - P[B] + P[A \cap B]$

2.25 $z = P[A \cup B] = P[A] + P[B] - P[A \cap B] = x + y - P[A \cap B]$



$P[A \cap B] = x + y - z$

$P[A^c \cap B^c] = 1 - P[(A^c \cap B^c)^c] = 1 - P[A \cup B]$

$= 1 - z$

$P[A^c \cup B^c] = 1 - P[(A^c \cup B^c)^c] = 1 - P[A \cap B] = 1 - x - y + z$

$P[A \cap B^c] = P[A] - P[A \cap B] = x - (x + y - z) = z - y$

$P[A^c \cup B] = 1 - P[A \cap B^c] = 1 - z + y$

2.26 Identities of this type are shown by application of the axioms. We begin by treating $(A \cup B)$ as a single event, then

$$\begin{aligned}
P[A \cup B \cup C] &= P[(A \cup B) \cup C] \\
&= P[A \cup B] + P[C] - P[(A \cup B) \cap C] && \text{by Cor. 5} \\
&= P[A] + P[B] - P[A \cap B] + P[C] && \text{by Cor. 5 on } A \cup B \\
&\quad - P[(A \cap C) \cup (B \cap C)] && \text{and by distributive property} \\
&= P[A] + P[B] + P[C] - P[A \cap B] \\
&\quad - P[A \cap C] - P[B \cap C] && \text{by Cor. 5 on} \\
&\quad + P[(A \cap B) \cap (B \cap C)] && (A \cap C) \cup (B \cap C) \\
&= P[A] + P[B] + P[C] - P[A \cap B] - P[A \cap C] && \text{since} \\
&\quad - P[B \cap C] + P[A \cap B \cap C]. && (A \cap B) \cap (B \cap C) = A \cap B \cap C
\end{aligned}$$

2.35 Assume that the probability of any subinterval I of $[-1,1]$ is proportional to its length, then

$$P[I] = k \text{ length } (I).$$

If we let $I = [-1,1]$ then we must have that

$$1 = P[S] = P[[-1,1]] = k \text{ length } ([-1,1]) = 2k \Rightarrow k = \frac{1}{2}.$$

a) $P[A] = \frac{1}{2} \text{ length } ([-1,0)) = \frac{1}{2}(1) = \frac{1}{2}$

$P[B] = \frac{1}{2} \text{ length } ((-0.5,1)) = \frac{1}{2}\frac{3}{2} = \frac{3}{4}$

$P[C] = \frac{1}{2} \text{ length } ((0.75,1)) = \frac{1}{2}\frac{1}{4} = \frac{1}{8}$

$P[A \cap B] = \frac{1}{2} \text{ length } ((-0.5,0)) = \frac{1}{2}\frac{1}{2} = \frac{1}{4}$

$P[A \cap C] = P[\emptyset] = 0$

b) $P[A \cup B] = P[\mathcal{S}] = 1$

$$\begin{aligned} P[A \cup C] &= \frac{1}{2} \text{ length } (A \cup C) \\ &= \frac{1}{2}\left(1 + \frac{1}{4}\right) = \frac{5}{8} \end{aligned}$$

$P[A \cup B \cup C] = P[\mathcal{S}] = 1$

Now use axioms and corollaries:

$$\begin{aligned} P[A \cup B] &= P[A] + P[B] - P[A \cap B] \quad \text{by Cor. 5} \\ &= \frac{1}{2} + \frac{3}{4} - \frac{1}{4} = 1 \quad \surd \\ P[A \cap C] &= P[A] + P[C] - P[\underbrace{A \cap C}_{\emptyset}] = \frac{1}{2} + \frac{1}{8} = \frac{5}{8} \quad \surd \quad \text{by Cor. 5} \\ P[A \cup B \cup C] &= P[A] + P[B] + P[C] \\ &\quad -P[A \cap B] - P[A \cap C] - P[B \cap C] \\ &\quad +P[A \cap B \cap C] \quad \text{by Eq. (2.11)} \\ &= \frac{1}{2} + \frac{3}{4} + \frac{1}{8} - \frac{1}{4} - 0 - \frac{1}{8} + 0 \\ &= 1 \quad \surd \end{aligned}$$

2.38 a) Since $(-\infty, r] \subset (-\infty, s]$ when $r < s$

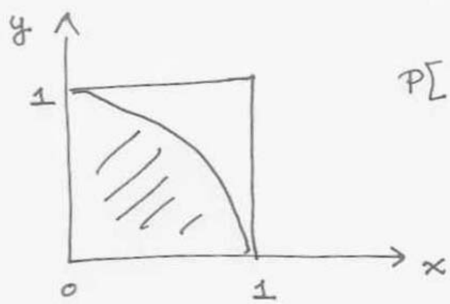
$$P[(-\infty, r]] \leq P[(-\infty, s]] \text{ by Corollary 7.}$$

b)

$$\begin{aligned} P[(-\infty, s]] &= P[(-\infty, r] \cup (r, s]] \\ &= P[(-\infty, r]] + P[(r, s]] \\ \Rightarrow P[(r, s]] &= P[(-\infty, s]] - P[(-\infty, r]] \end{aligned}$$

2.39

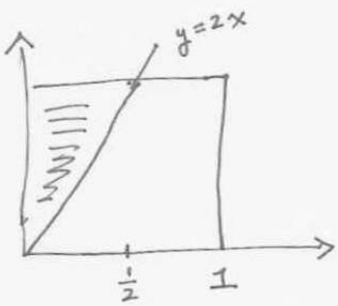
(a)



$P[x^2+y^2 < 1] = \frac{\pi(1)^2}{4} = \frac{\pi}{4}$

Area inside circle

(b)



$P[y > 2x] = \frac{1}{4}$

Area in right triangle

2.56 Number ways of picking 20 raccoons out of $N = \binom{N}{20}$

Number ways of picking $\cancel{5}\ 4$ tagged raccoons out of 10 and $\cancel{15}\ 16$ untagged raccoons out of $N-10$ $= \binom{10}{\cancel{5}\,4}\binom{N-10}{\cancel{15}\,16}$

$$P[\cancel{5}\,4 \text{ tagged out of 20 samples}] = \frac{\binom{10}{\cancel{5}\,4}\binom{N-10}{\cancel{15}\,16}}{\binom{N}{20}} \triangleq p(N)$$

$p(N)$ increases with N as long as $p(N)/p(N-1) > 1$

$$\frac{p(N)}{p(N-1)} = \frac{\binom{N-10}{\cancel{15}\,16}\binom{N-1}{20}}{\binom{N}{20}\binom{N-11}{\cancel{15}\,16}} = \frac{(N-10)(N-20)}{(N-\cancel{25}\,26)N} \geq 1$$

$(N-10)(N-20) \geq (N-\cancel{25}\,26)N \Rightarrow \cancel{40}\,50 \geq N$

$p(\cancel{40}\,50) = p(\cancel{39}\,49) = \cancel{.284}\,.280$ maxima of $p(N)$.

2.57

(b) $P[X=k] = \dfrac{\binom{10}{k}\binom{40}{5-k}}{\binom{50}{5}}$ $k=0,1,\ldots,5$ without replacement

Hypergeometric probabilities

(a) With replacement:

pick k defective balls then pick $5-k$ nondefective balls

$\underbrace{\text{pick } k \text{ defective balls}}_{10^k}$ $\underbrace{\text{then pick } 5-k \text{ nondefective balls}}_{40^{5-k}}$

There are $\binom{5}{k}$ arrangements of this composition

\# ways of obtaining k defective in 5 tested $= \dfrac{\binom{5}{k}10^k 40^{5-k}}{50^5}$

$= \binom{5}{k}\left(\frac{10}{50}\right)^k\left(\frac{40}{50}\right)^{5-k}$ $k=0,1,\ldots,5$

Binomial probabilities.

2.69

$$P[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{P[\Phi]}{P[(0,1)]} = 0$$

$$P[B|C] = \frac{P[(0,1) \cap (.75,2]]}{P[(.75,2]]} = \frac{P[(.75,1)]}{P[(.75,2]]} = \frac{\frac{1}{4}(1/4)}{\frac{1}{4}(5/4)} = \frac{1}{5}$$

$$P[A|C^c] = \frac{P[[-2,0) \cap [-2,.75]]}{P[[-2,.75]]} = \frac{P[[-2,0)]}{P[[-2,.75]]} = \frac{\frac{1}{4}\cdot 2}{\frac{1}{4}\,2.75} = \frac{8/4}{11/4} = \frac{8}{11}$$

$$P[B|C^c] = \frac{P[(0,1) \cap [-2,.75]]}{P[[-2,.75]]} = \frac{P[(0,.75)]}{P[[-2,.75]]} = \frac{\frac{1}{4}\cdot\frac{3}{4}}{\frac{1}{4}\,\frac{11}{4}} = \frac{3}{11}$$

2.70

$$P[x > 2t / x > t] = \frac{P[\{x>2t\} \cap \{x>t\}]}{P[x>t]} = \frac{P[x>2t]}{P[x>t]}$$

$$= \frac{1/2t}{1/t} = \frac{1}{2} \qquad t > 1$$

This conditional probability does not depend on t.
The corresponding probability law is said to be scale-invariant.

2.71

P[2 or more students have some birthday]

$$= 1 - P[\text{all students have different birthdays}]$$

P[all students have different birthdays]

$$= \frac{365}{365}\frac{364}{365}\frac{363}{365}\cdots\frac{346}{365} = 0.588$$

P[2 or more have same birthday] = 0.412

P[2 or more have same birthday in class of 23] $= 0.507$

2.73 a) The results follow directly from the definition of conditional probability. $P[A|B] = \frac{P[A \cap B]}{P[B]}$

If $A \cap B = \emptyset$ then $P[A \cap B] = 0$ by Corollary 3 and thus $P[A|B] = 0$.

If $A \subset B$ then $A \cap B = A$ and $P[A|B] = \frac{P[A]}{P[B]}$.

If $A \supset B \Rightarrow A \cap B = B$ and $P[A|B] = \frac{P[B]}{P[B]} = 1$.

b) If $P[A|B] = \frac{P[A \cap B]}{P[B]} > P[A]$ then multiplying both sides by $P[B]$ we have: $P[A \cap B] > P[A]P[B]$

We then also have that $P[B|A] = \frac{P[A \cap B]}{P[A]} > \frac{P[A]P[B]}{P[A]} = P[B]$.

We conclude that if $P[A|B] > P[A]$ then B and A tend to occur jointly.

2.74

$$P[A|B] = \frac{P[A \cap B]}{P[B]} \quad \text{for } P[B] > 0.$$

(i) $P[A \cap B] \geq 0 \Rightarrow P[A|B] \geq 0.$ ✓

$A \cap B \subset B \Rightarrow P[A \cap B] \leq P[B] \Rightarrow P[A|B] \leq 1.$ ✓

(ii) $P[S|B] = \frac{P[B \cap S]}{P[B]} = \frac{P[B]}{P[B]} = 1$ ✓

(iii) If $A \cap C = \phi$ then

$$P[A \cup C|B] = \frac{P[(A \cup C) \cap B]}{P[B]} = \frac{P[(A \cap B) \cup (C \cap B)]}{P[B]}$$

$$= \frac{P[A \cap B] + P[C \cap B]}{P[B]} \quad \text{since } (A \cap B) \cap (C \cap B) = A \cap B \cap C = \phi$$

$$= P[A|B] + P[C|B]$$ ✓

2.75

$$\begin{aligned} P[A \cap B \cap C] &= P[A|B \cap C]P[B \cap C] \\ &= P[A|B \cap C]P[B|C]P[C] \end{aligned}$$

2.77 Let X denote the input and Y the output

(a) $P[Y=0] = P[Y=0 \mid X=0]\,P[X=0] + P[Y=0 \mid X=1]\,P[X=1]$

$$= (1-\varepsilon_1)\,q + \varepsilon_2\,p.$$

Similarly

$$P[Y=1] = (1-\varepsilon_2)\,p + \varepsilon_1\,q$$

(b) $$P[X=0 \mid Y=1] = \frac{P[Y=1 \mid X=0]\,P[X=0]}{P[Y=1]} = \frac{\varepsilon_1 q}{(1-\varepsilon_2)p + \varepsilon_1 q}$$

$$P[X=1 \mid Y=1] = \frac{(1-\varepsilon_2)\,p}{(1-\varepsilon_2)p + \varepsilon_1 q}$$

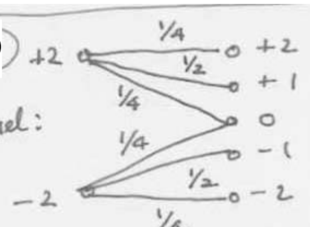
$$P[X=1 \mid Y=1] > P[X=0 \mid Y=1]$$

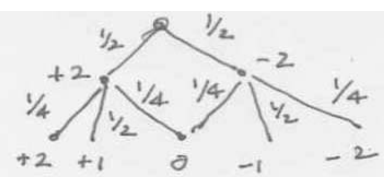
$$\Leftrightarrow \quad (1-\varepsilon_2)\,p > \varepsilon_1 q = \varepsilon_1(1-p)$$

$$\Leftrightarrow \quad p > \frac{\varepsilon_1}{1-\varepsilon_2+\varepsilon_1}$$

2.78

channel:





(b) $P[X=+2, Y=+2] = P[Y=2 \mid X=2]\,P[X=2]$

$$= \tfrac{1}{4}\cdot\tfrac{1}{2} = \tfrac{1}{8}$$

$P[X=2, Y=1] = \frac{1}{2}\frac{1}{2} = \frac{1}{4}$

$P[X=2, Y=0] = \frac{1}{2}\frac{1}{4} = \frac{1}{8}$

$P[X=-2, Y=0] = \frac{1}{2}\frac{1}{4} = \frac{1}{8}$

$P[X=-2, Y=1] = \frac{1}{2}\frac{1}{2} = \frac{1}{4}$

$P[X=-2, Y=-2] = \frac{1}{2}\frac{1}{4} = \frac{1}{8}$

(c) $P[Y=+2] = \frac{1}{2}\cdot\frac{1}{4} = \frac{1}{8} = P[Y=-2]$

$P[Y=+1] = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4} = P[Y=-1]$

$P[Y=0] = 2\left(\frac{1}{2}\cdot\frac{1}{4}\right) = \frac{2}{8} = P[Y=0]$

(d) $$P[X=2 \mid Y=k] = \frac{P[Y=k \mid X=2]\,P[X=2]}{P[Y=k]}$$

$$= \begin{cases} \frac{1}{8}\big/\frac{1}{8} = 1 & k=2 \\ \frac{1}{4}\big/\frac{1}{4} = 1 & k=1 \\ \frac{1}{8}\big/\frac{1}{4} = \frac{1}{2} & k=0 \\ 0 & \text{other } k \end{cases}$$

2.80

$$\begin{aligned} P[\text{chip defective}] &= P[\text{def.}|A]P[A] + P[\text{def.}|B]P[B] + P[\text{def,}|C]P[C] \\ &= (10^{-3})p_A + 5(10^{-3})p_B + 10(10^{-3})p_C \end{aligned}$$

$$\begin{aligned} P[A|\text{chip defective}] &= \frac{P[\text{def.}|A]P[A]}{P[\text{def.}]} = \frac{10^{-3}p_A}{10^{-3}p_A + 5(10^{-3})p_B + 10(10^{-3})p_C} \\ &= \frac{p_A}{p_A + 5p_B + 10p_C} \end{aligned}$$

Similarly

$$P[C|\text{chip defective}] = \frac{10p_C}{p_A + 5p_B + 10p_C}$$

2.81 Let X denote the input and Y the output.

a)
$$\begin{aligned} P[Y=0] &= P[Y=0|X=0]P[X=0] + P[Y=0|X=1]P[X=0] \\ &\quad + P[Y=0|X=2]P[X=2] \\ &= (1-\varepsilon)\tfrac{1}{2} +) \cdot \tfrac{1}{4} + \varepsilon \cdot \tfrac{1}{4} \\ &= \tfrac{1}{2} - \tfrac{\varepsilon}{4} \end{aligned}$$

Similarly

$$\begin{aligned} P[Y=1] &= \varepsilon \cdot \frac{1}{2} + (1-\varepsilon)\frac{1}{4} + 0\varepsilon\frac{1}{4} = \frac{1}{4} + \frac{\varepsilon}{4} \\ P[Y=2] &= 0 \cdot \frac{1}{2} + \varepsilon \cdot \frac{1}{4} + (1-\varepsilon)\frac{1}{4} = \frac{1}{4} \end{aligned}$$

b) Using Bayes' Rule

$$\begin{aligned} P[X=0|Y=1] &= \frac{P[Y=1|X=0]P[X=0]}{P[Y=1]} = \frac{\frac{1}{2}\varepsilon}{\frac{1}{4} + \frac{\varepsilon}{4}} = \frac{2\varepsilon}{1+\varepsilon} \\ P[X=1|Y=1] &= \frac{P[Y=1|X=1]P[X=1]}{P[Y=1]} = \frac{(1-\varepsilon)\frac{1}{4}}{\frac{1}{4} + \frac{\varepsilon}{4}} = \frac{1-\varepsilon}{1+\varepsilon} \\ P[X=2|Y=1] &= 0 \end{aligned}$$

2.5 Independence of Events

2.82

$P[A \cap B] = P[\{1\}] = \frac{1}{4} = P[A]P[B] = \frac{1}{2}\frac{1}{2}$ ✓

$P[A \cap C] = P[\{1\}] = \frac{1}{4} = P[A]P[C] = \frac{1}{2}\frac{1}{2}$ ✓

$P[B \cap C] = P[\{1\}] = \frac{1}{4} = P[B]P[C] = \frac{1}{2}\frac{1}{2}$ ✓

$P[A \cap B \cap C] = P[\{1\}] = \frac{1}{4} \neq P[A]P[B]P[C] = \frac{1}{2}\frac{1}{2}\frac{1}{2} = \frac{1}{8}$

$\Rightarrow$ Not Independent

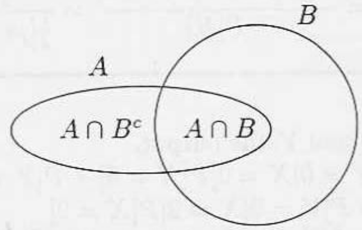
2.83

$P[A \cap B] = P[\frac{1}{4} < U < \frac{1}{2}] = \frac{1}{4} = P[A]P[B] = \frac{1}{2}\frac{1}{2}$ ✓ A & B indep

$P[A \cap C] = 0 \neq P[A]P[C] = \frac{1}{2}\frac{1}{2} \Rightarrow$ A & C Not Indep.

$P[B \cap C] = P[\frac{1}{2} < U < \frac{3}{4}] = \frac{1}{4} = P[B]P[C] = \frac{1}{2} \cdot \frac{1}{2}$ ✓ B & C indep.

2.84 The event A is the union of the mutually exclusive events $A \cap B$ and $A \cap B^c$, thus



$$\begin{aligned}
P[A] &= P[A \cap B] + P[A \cap B^c] && \text{by Corollary 1} \\
\Longrightarrow P[A \cap B^c] &= P[A] - P[A \cap B] \\
&= P[A] - P[A]P[B] && \text{since } A \text{ and } B \text{ are independent} \\
&= P[A](1 - P[B]) \\
&= P[A]P[B^c] \Longrightarrow && A \text{ and } B^c \text{ are independent}
\end{aligned}$$

Similarly

$$P[B] = P[A \cap B] + P[A^c \cap B] = P[A]P[A] + P[A^c \cap B]$$

$$\begin{aligned}
\Longrightarrow P[A^c \cap B] = P[B](1 - P[A]) &= P[B]P[A^c] \\
&\Rightarrow A \text{ and } B \text{ are independent}
\end{aligned}$$

Finally

$$P[A^c] = P[A^c \cap B] + P[A^c \cap B^c] = P[A^c]P[B] + P[A^c \cap B^c]$$

$$\begin{aligned}
\Longrightarrow P[A^c \cap B^c] = P[A^c](1 - P[B]) &= P[A^c]P[B^c] \\
&\Rightarrow A^c \text{ and } B^c \text{ are independent}
\end{aligned}$$

2.85

$$P[A|B] = P[A|B^c] \Longrightarrow \frac{P[A \cap B]}{P[B]} = \frac{P[A \cap B^c]}{P[B^c]}$$

$$\begin{aligned}
\Longrightarrow P[A \cap B]P[B^c] &= P[A \cap B^c]P[B] \\
&= (P[A] - P[A \cap B])P[B] && \text{see Prob. 2.58 solution}
\end{aligned}$$

$$\Longrightarrow P[A \cap B]\underbrace{(P[B^c] + P[B])}_{1} = P[A]P[B]$$

$$\Longrightarrow P[A \cap B] = P[A]P[B]$$